INVERSE PROBLEMS OF PARAMETER AND FUNCTION ESTIMATION

HELCIO R. B. ORLANDE
Department of Mechanical Engineering, DEM/PEM
Escola Politécnica/COPPE
Federal University of Rio de Janeiro, UFRJ
Cid. Universitária, Cx. Postal: 68503
Rio de Janeiro, RJ, 21941-972, Brazil
helcio@mecanica.coppe.ufrj.br

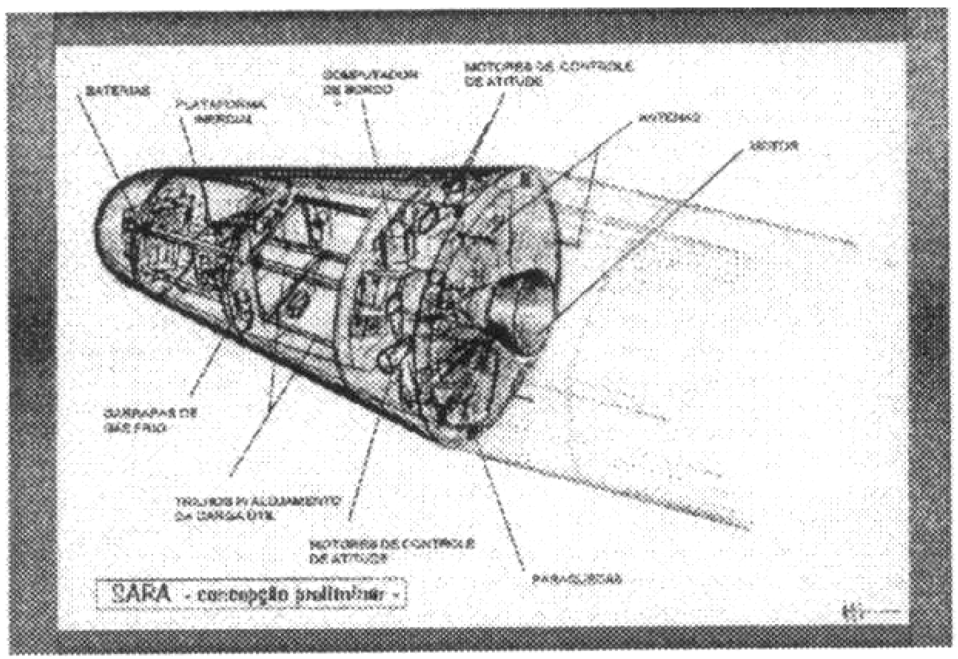
Engenharia
Mecânica
Politécnica ▪ COPPE
UFRJ

OUTLINE

- Motivation
- Physical Problem and Mathematical Formulation
- Direct Problem and Inverse Problem
- Methods of Solution for the Inverse Problem
  - Parameter Estimation
  - Function Estimation
- Experimental Setup
- Results and Discussions
- Conclusions

1. MOTIVATION

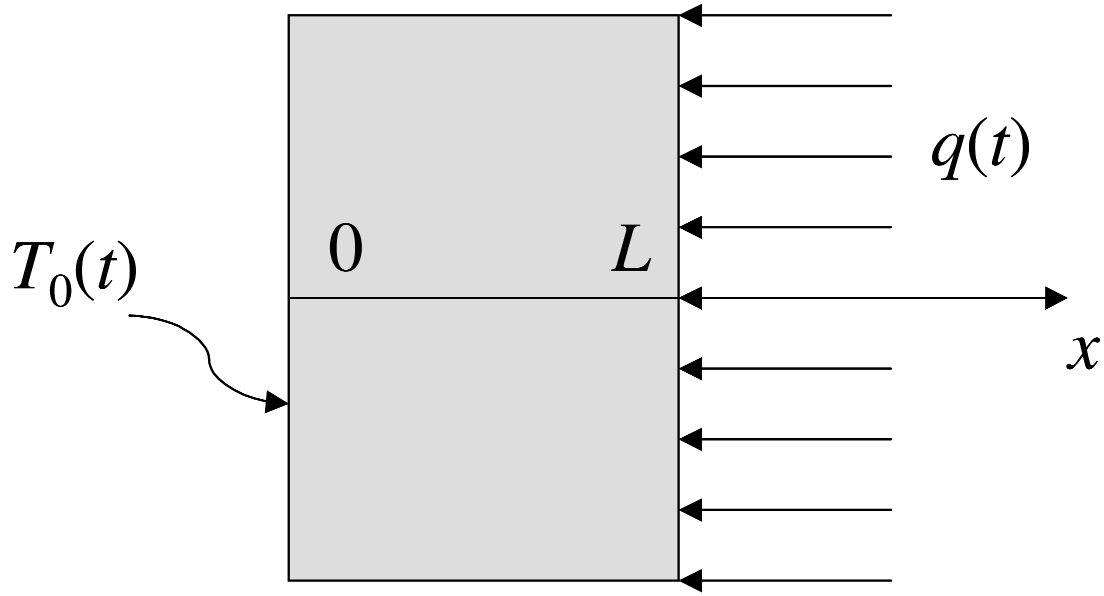
- Design the heat-shield for *SARA* Satellite - Brazilian Space Agency



1. MOTIVATION

- Tragic accident with the Brazilian Satellite Launcher Vehicle
- August 22, 2003
- 21 Engineers and Technicians dead

2. PHYSICAL PROBLEM AND MATHEMATICAL FORMULATION



2. PHYSICAL PROBLEM AND MATHEMATICAL FORMULATION

$$C(T)\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] \quad \text{in} \quad 0 < x < L,\ t > 0$$

$$T = T_0(t) \quad \text{at} \quad x = 0, \quad t > 0$$

$$k(T)\frac{\partial T}{\partial x} = q(t) \quad \text{at} \quad x = L, \quad t > 0$$

$$T = T_{ini} \quad \text{for } t = 0, \text{ in} \quad 0 < x < L$$

3. DIRECT PROBLEM AND INVERSE PROBLEM

DIRECT PROBLEM

Known

• Boundary and initial conditions

• Thermophysical properties

↓

Determine

• Temperature distribution

$T(x,t)$

INVERSE PROBLEM

Known

• Initial condition

• Boundary condition at $x = 0$

• ***Thermophysical properties***

• ***Temperature measurements***

↓

Estimate

$q(t)$

PARAMETER ESTIMATION

The unknown function $q(t)$ is approximated as:

$$q(t) = \sum_{j=1}^{N} P_j C_j(t)$$

where:

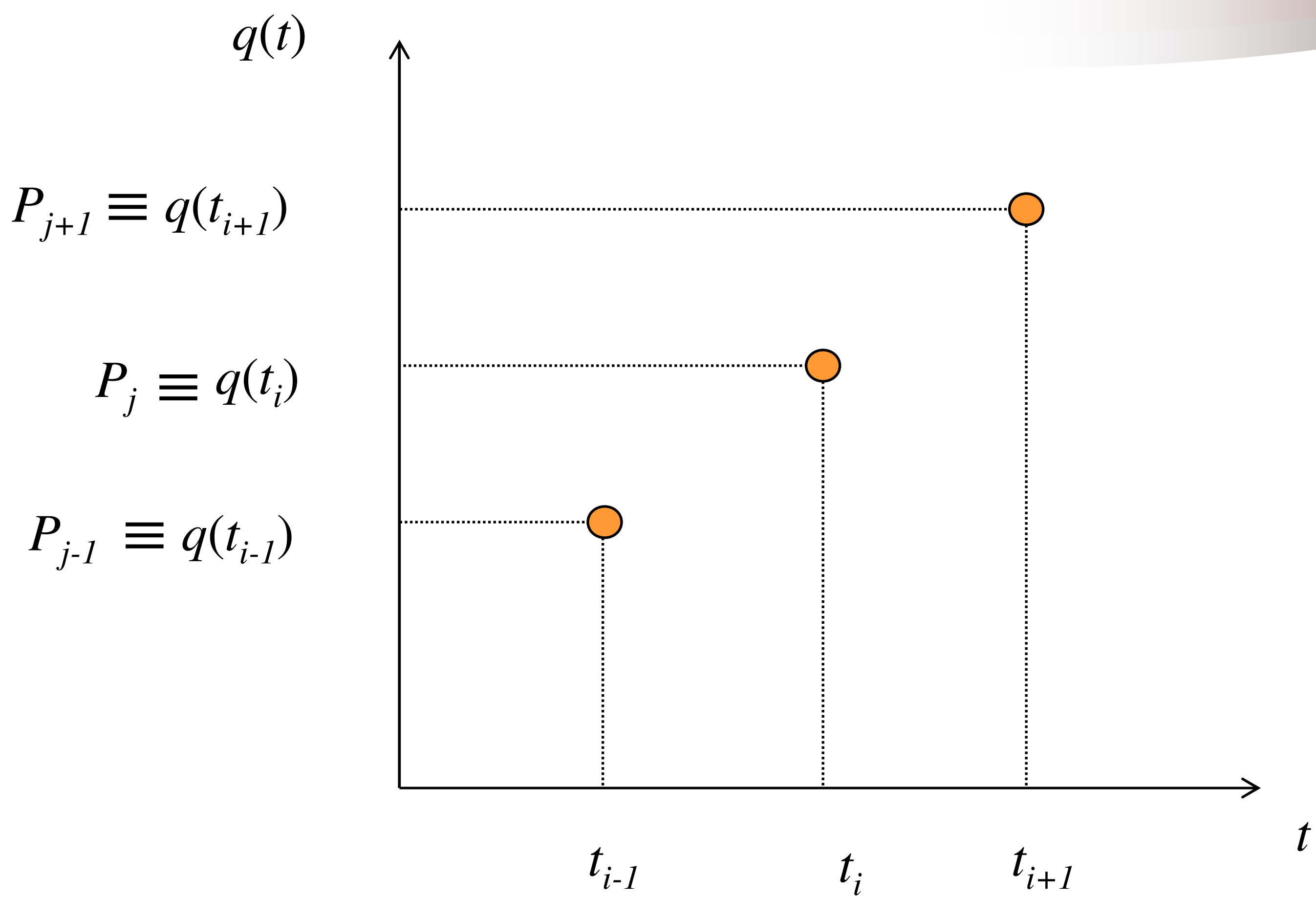
$C_j(t)$ are *known basis functions*
N is the *number of basis functions* used in the approximation (known for the analysis)
P_j are the *unknown* parameters

FUNCTION ESTIMATION

- No assumption is made regarding the functional form of the unknown.
- Minimization is performed in an infinite dimensional space of functions, or minimization is performed in a finite dimensional space where N is large, e.g., $C_j(t) = \delta(t_i)$, $i = 1,\ldots,I$, $N = I$.

I = Number of measurements
N = Number of unknown parameters

FUNCTION ESTIMATION



AN OVERVIEW OF SOLUTION TECHNIQUES FOR INVERSE HEAT TRANSFER PROBLEMS

Remark: If the inverse heat transfer problem involves the estimation of only few unknown parameters from transient temperature measurements, the use of the ordinary least squares norm can be stable. However, if the inverse problem involves the estimation of a large number of parameters, such as the recovery of the unknown transient heat flux $q(t_i)$ at times t_i, $i=1,\ldots,I$, excursion and oscillation of the solution may occur. In this case, ***regularization*** (or stabilization) techniques are required.

Tikhonov's Whole-Domain Regularization

Zeroth order:

$$S[g_p(t)] = \sum_{i=1}^{I} (Y_i - T_i)^2 + \alpha_0 \sum_{i=1}^{I} [g_p(t_i)]^2$$

First order:

$$S[g_p(t)] = \sum_{i=1}^{I} (Y_i - T_i)^2 + \alpha_1 \sum_{i=1}^{I-1} [g_p(t_{i+1}) - g_p(t_i)]^2$$

- α_0 and α_1 are the regularization parameters.

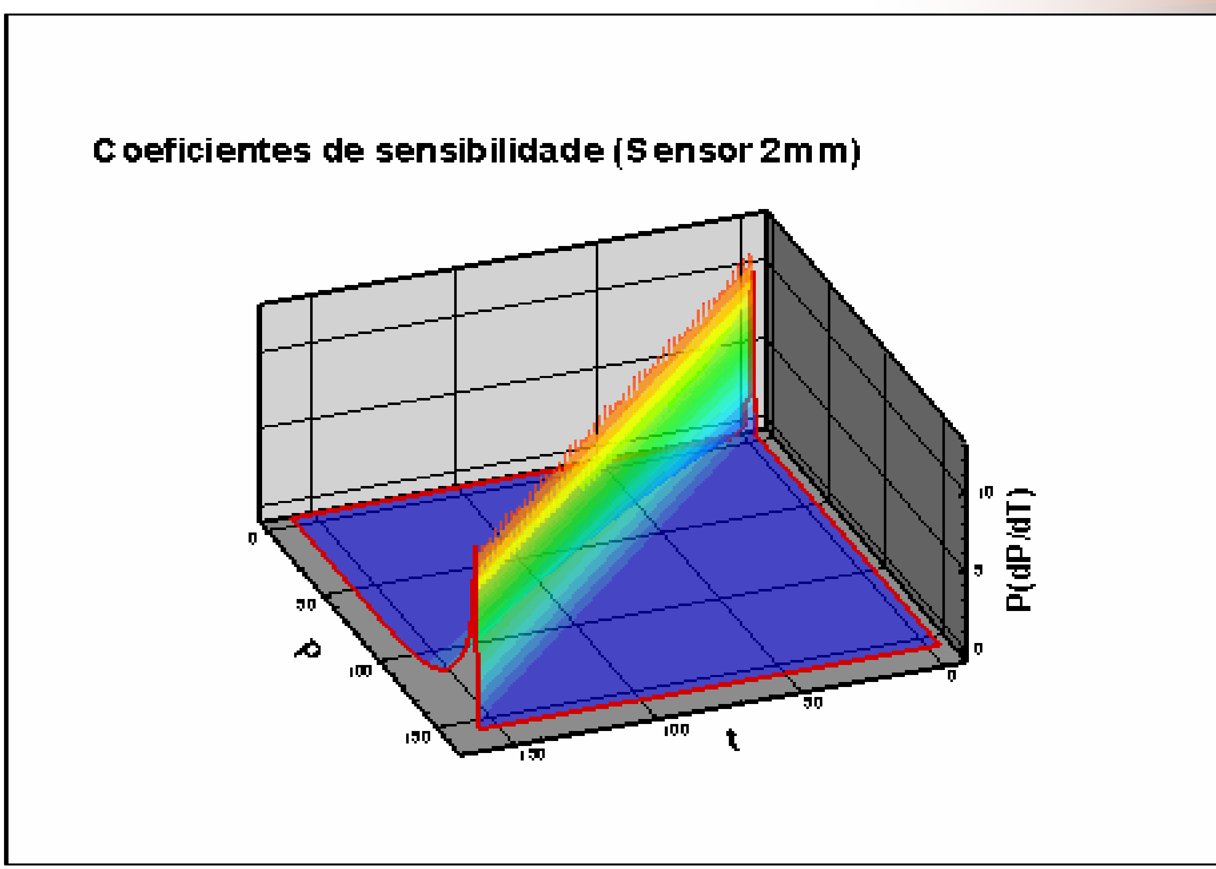
Beck's Sequential Function Specification Technique

$$S[q(t_i)] = \sum_{s=i}^{i+r-1} (Y_s - T_s)^2$$

where r is the number of future measurements

- *Regularization* is obtained from the least-squares averaging capabilities and from the measurements taken at future time steps.

AN OVERVIEW OF SOLUTION TECHNIQUES FOR INVERSE HEAT TRANSFER PROBLEMS



Alifanov’s Iterative Regularization

- Regularization is obtained from the stopping criterion utilized for the iterative procedure.

4. METHOS OF SOLUTION FOR THE INVERSE PROBLEM

PARAMETER ESTIMATION

- Constant Heat Flux: $q(t) = q_0$
- Requires accurate knowledge of initial time
- Statistical Hypotheses:

  - measurement errors are additive, uncorrelated, normally distributed, with zero mean and known constant standard-deviation;
  - only the measured variables contain errors;
  - ***there is no prior information regarding the values and uncertainties of the unknown parameters***

4. METHOS OF SOLUTION FOR THE INVERSE PROBLEM

PARAMETER ESTIMATION

Minimization of: $$S_{OLS}(\mathbf{P}) = [\mathbf{Y} - \mathbf{T}(\mathbf{P})]^T [\mathbf{Y} - \mathbf{T}(\mathbf{P})]$$

Levenberg-Marquardt's Method

$$\mathbf{P}^{k+1} = \mathbf{P}^k + (\mathbf{J}^T \mathbf{J} + \mu^k \,\Omega^k)^{-1} \mathbf{J}^T [\mathbf{Y} - \mathbf{T}(\mathbf{P}^k)]$$

where: $\mathbf{P} = [q_0]$

$\mathbf{J}$ is the sensitivity matrix,

$\mathbf{\Omega}^k$ is a diagonal matrix

μ^k is a scalar named damping parameter

SENSITIVITY MATRIX AND SENSITIVITY COEFFICIENTS

$$\mathbf{J} = \left[\frac{\partial \mathbf{T}^T}{\partial \mathbf{P}}\right]^T = \begin{bmatrix} \frac{\partial T_1}{\partial P_1} & \frac{\partial T_1}{\partial P_2} & \frac{\partial T_1}{\partial P_3} & \cdots & \frac{\partial T_1}{\partial P_N} \\ \frac{\partial T_2}{\partial P_1} & \frac{\partial T_2}{\partial P_2} & \frac{\partial T_2}{\partial P_3} & \cdots & \frac{\partial T_2}{\partial P_N} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial T_I}{\partial P_1} & \frac{\partial T_I}{\partial P_2} & \frac{\partial T_I}{\partial P_3} & \cdots & \frac{\partial T_I}{\partial P_N} \end{bmatrix}$$

$$J_{ij} = \frac{\partial T_i}{\partial P_j}$$

4. METHOS OF SOLUTION FOR THE INVERSE PROBLEM

FUNCTION ESTIMATION

- No *a priori* assumption regarding the functional form of the unknown.
- Hilbert space of square integrable functions in the domain $0 < t < t_f$:

$$\int_{t=0}^{t_f} [q(t)]^2 dt < \infty$$

Minimization of:

$$S[q(t)] = \frac{1}{2} \sum_{m=1}^{M} \int_{t=0}^{t_f} \{T_m[t;q(t)] - Y_m(t)\}^2 dt$$

where: $Y_m(t)$ = measured temperatures

$T_m[t;q(t)]$ = estimated temperatures

M = number of sensors

t_f = final time

CONJUGATE GRADIENT METHOD

Iterative Procedure: $q^{k+1}(t) = q^k(t) - \beta^k d^k(t)$

Direction of Descent: $d^k(t) = \nabla S[q^k(t)] + \gamma^k d^{k-1}(t)$

Conjugation Coefficient:
(Fletcher-Reeves)

$$\gamma^k = \frac{\int_{t=0}^{t_f} \{\nabla S[q^k(t)]\}^2 dt}{\int_{t=0}^{t_f} \{\nabla S[q^{k-1}(t)]\}^2 dt}$$

Search step size β^k
and
Gradient Direction $\nabla S[q(t)]$

→

Sensitivity Problem
and
Adjoint Problem

SENSITIVITY PROBLEM

$q(t)$ is perturbed by $\varepsilon\Delta q(t)$ → $T(x,t)$ undergoes a variation $\varepsilon\Delta T(x,t)$

Because of non-linearities:

$$k(T_{\varepsilon}) = k(T + \varepsilon\Delta T) \approx k(T) + \left(\frac{dk}{dT}\right)\varepsilon\Delta T$$

$$C(T_{\varepsilon}) = C(T + \varepsilon\Delta T) \approx C(T) + \left(\frac{dC}{dT}\right)\varepsilon\Delta T$$

SENSITIVITY PROBLEM

$$\lim_{\varepsilon \to 0} \frac{D[q_{\varepsilon}(t)] - D[q(t)]}{\varepsilon} = 0$$

where $D[q_{\varepsilon}(t)]$ is the operator form of the direct problem written for the perturbed quantities.

$$\frac{\partial (C\Delta T)}{\partial t} - \frac{\partial^2 (k\Delta T)}{\partial x^2} = 0 \qquad \text{in } 0 < x < L, \text{ for } t > 0$$

$$\Delta T = 0 \qquad \text{at } x = 0, \text{ for } t > 0$$

$$\frac{\partial (k\Delta T)}{\partial x} = \Delta q(t) \qquad \text{at } x = L, \text{ for } t > 0$$

$$\Delta T = 0 \qquad \text{for } t = 0, \text{ in } 0 < x < L$$

SEARCH STEP SIZE

$$\min_{\beta^k} S[q^{k+1}(t)] = \min_{\beta^k} \frac{1}{2} \sum_{m=1}^{M} \int_{t=0}^{t_f} \left\{ T\left[x_m, t; q^k(t) - \beta^k d^k(t)\right] - Y_m(t) \right\}^2 dt$$

$$\beta^k = \frac{\displaystyle\sum_{m=1}^{M} \int_{t=0}^{t_f} \{T[x_m, t; q(t)] - Y_m(t)\} \, \Delta T[x_m, t; d^k(t)] \, dt}{\displaystyle\sum_{m=1}^{M} \int_{t=0}^{t_f} \{\Delta T[x_m, t; d^k(t)]\}^2 \, dt}$$

where $\Delta T[x_m, t; d^k(t)]$ is the solution of the sensitivity problem obtained by setting $\Delta q(t) = d^k(t)$.

ADJOINT PROBLEM

Lagrange multiplier $\lambda(x,t)$

$$S[q(t)] = \frac{1}{2}\int_{x=0}^{L}\int_{t=0}^{t_f}\sum_{m=1}^{M}(T-Y)^2\,\delta(x-x_m)\,dt\,dx + \int_{x=0}^{L}\int_{t=0}^{t_f}\left\{C(T)\frac{\partial T(x,t)}{\partial t} - \frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right]\right\}\lambda(x,t)\,dt\,dx$$

ADJOINT PROBLEM

$$\Delta S[q(t)] = \lim_{\varepsilon \to 0} \frac{S[q_\varepsilon(t)] - S[q(t)]}{\varepsilon} = 0$$

$$\Delta S[q(t)] = \int_{t=0}^{t_f} \int_{x=0}^{L} \sum_{m=1}^{M} (T - Y)\delta(x - x_m)\Delta T \, dx \, dt + \int_{x=0}^{L} \int_{t=0}^{t_f} \frac{\partial (C\Delta T)}{\partial t} \lambda(x,t) \, dt dx -$$
$$- \int_{t=0}^{t_f} \int_{x=0}^{L} \frac{\partial^2 (k\Delta T)}{\partial x^2} \lambda(x,t) \, dx \, dt = 0$$

ADJOINT PROBLEM

$$-C\frac{\partial \lambda}{\partial t} - k\frac{\partial^2 \lambda}{\partial x^2} + \sum_{m=1}^{M}(T-Y)\delta(x-x_m) = 0 \qquad \text{in } 0 < x < L \text{, for } 0 < t < t_f$$

$$\lambda = 0 \qquad \text{at } x = 0, \text{for } 0 < t < t_f$$

$$\frac{\partial \lambda}{\partial x} = 0 \qquad \text{at } x = L, \text{for } 0 < t < t_f$$

$$\lambda = 0 \qquad \text{for } t = t_f \text{, in } 0 < x < L$$

GRADIENT EQUATION

$$\Delta S[q(t)] = -\int_{t=0}^{t_f} \lambda(L,t)\Delta q(t)\,dt \;\to 0$$

By invoking the hypothesis that $q(t)$ belongs to the L_2 space in $0 < t < t_f$:

$$\Delta S[q(t)] = \int_{t=0}^{t_f} \nabla S[q(t)]\Delta q(t)\,dt$$

Therefore:

$$\nabla S[q(t)] = -\lambda(L,t)$$

STOPPING CRITERION

$$S[q^{k+1}(t)] < \varepsilon$$

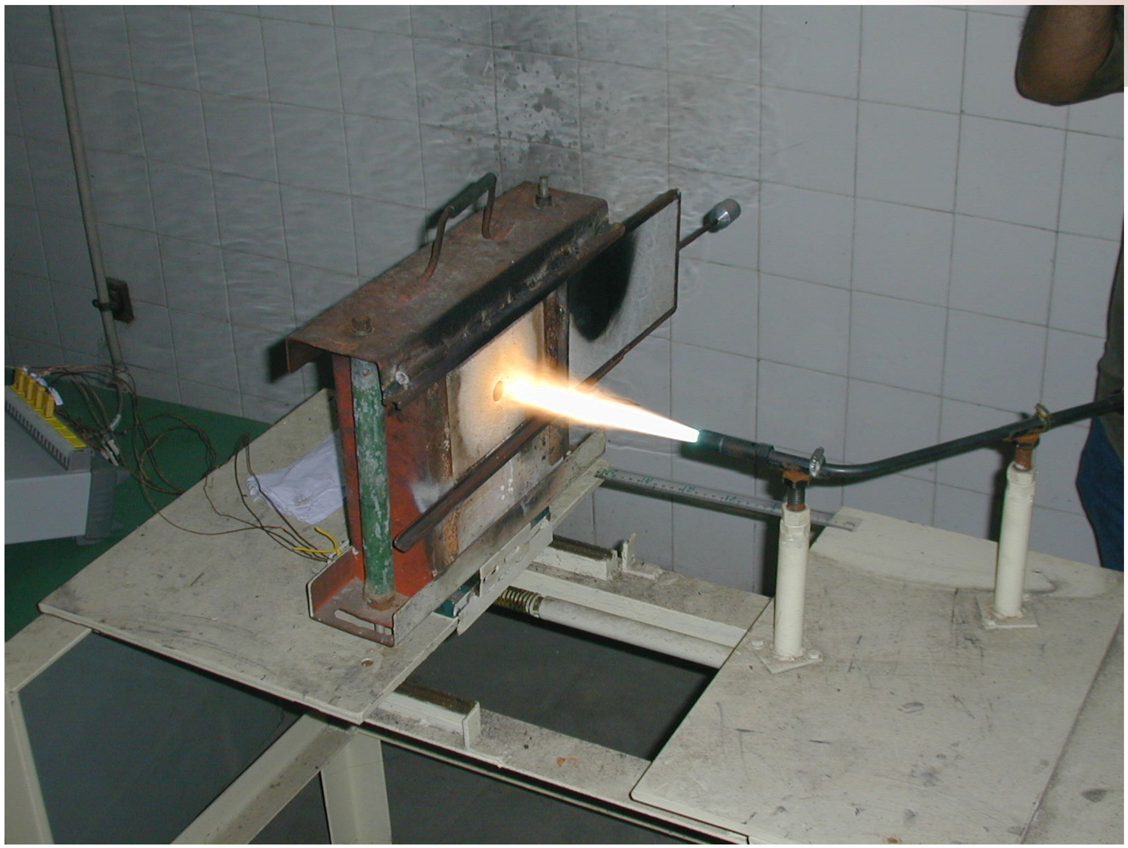
- Errorless measurements: ε is a small specified number
- Measurements containing random errors: Discrepancy Principle (iterative regularization)

$$|Y_m(t) - T[x_m, t; q(t)]| \approx \sigma$$

where σ = standard-deviation of the measurements

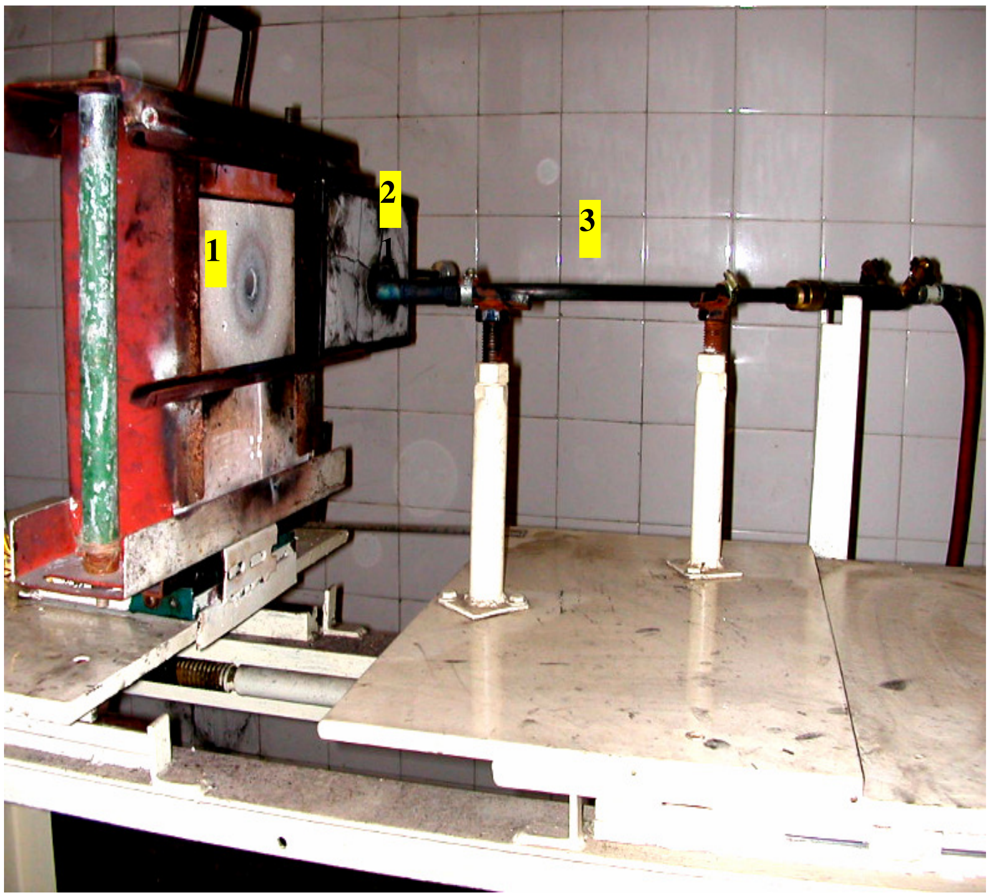
$$\varepsilon = \frac{1}{2} M \, \sigma^2 t_f$$

5. EXPERIMENTAL SETUP



5. EXPERIMENTAL SETUP

- ASTM standard E285-80



1 - Sample and refractory support

2 – Heat shield

3 – Oxy-acetylene torch

5. EXPERIMENTAL SETUP

- Cylindrical sample: Diameter and Thickness = 20 mm
- 20 scfh for oxygen and for acetylene
- 4 calibrated thermocouples type K, 30 gauge (0.25 mm diameter wires)

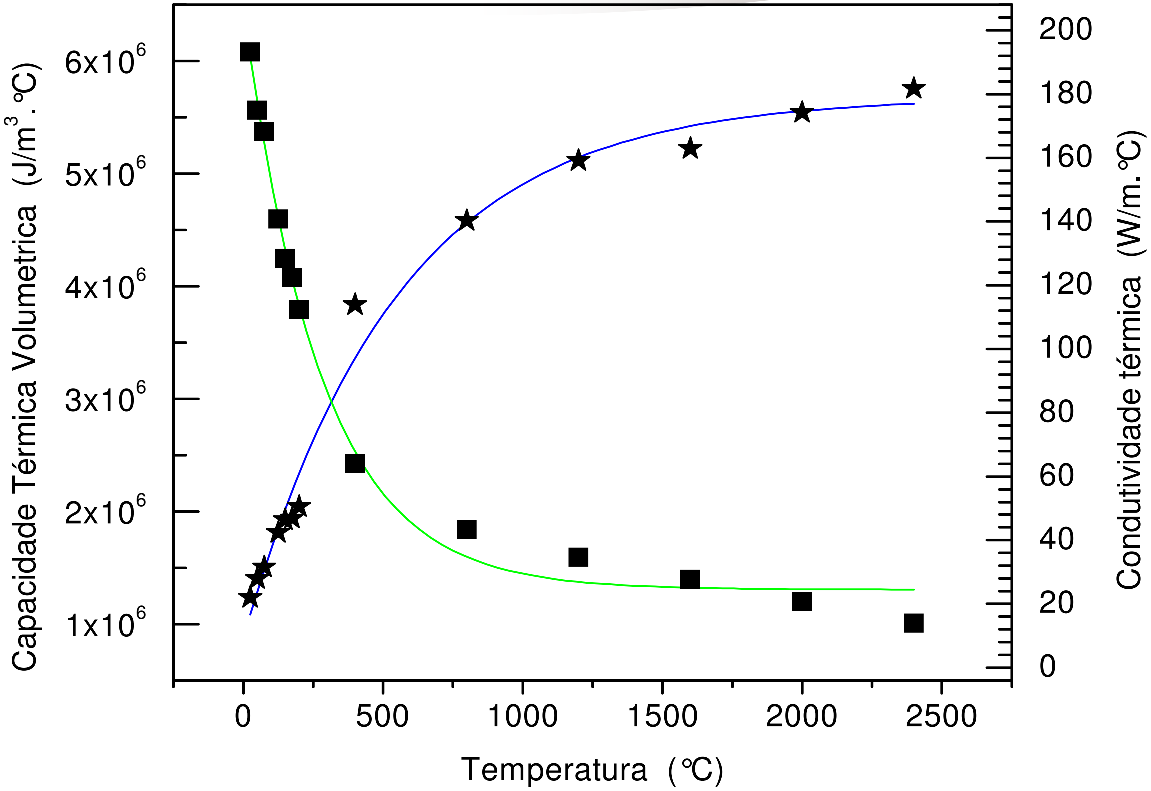
  Sensor 1: 2 mm below the heated surface

  Sensor 2: 5 mm below the heated surface

  Sensor 3: 10 mm below the heated surface

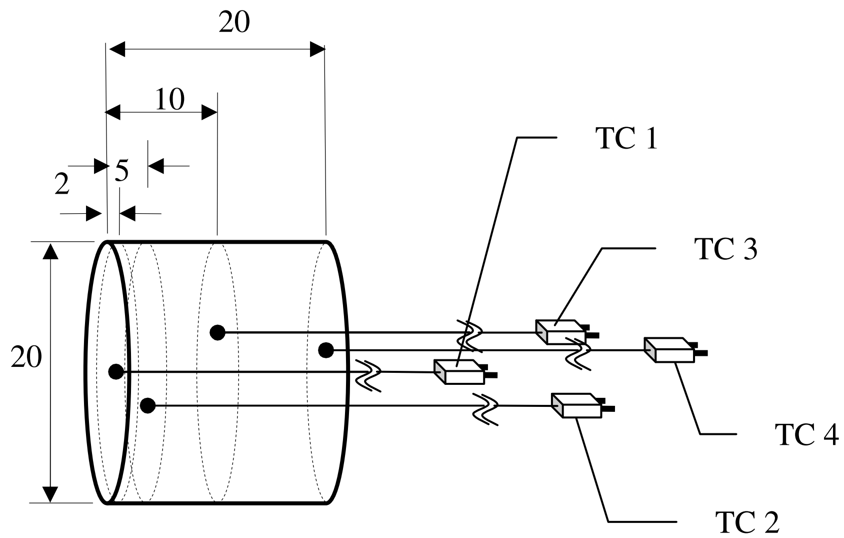
  Sensor 4: 20 mm below the heated surface (non-heated boundary)
- AGILENT 34970A data acquisition system: 1 measurement per second per sensor
- High-quality graphite used in rocket nozzles
- Graphite thermophysical properties from the manufacturer and from tests in a Flash Method Apparatus (NETZSCH LFA-447)

5. EXPERIMENTAL SETUP

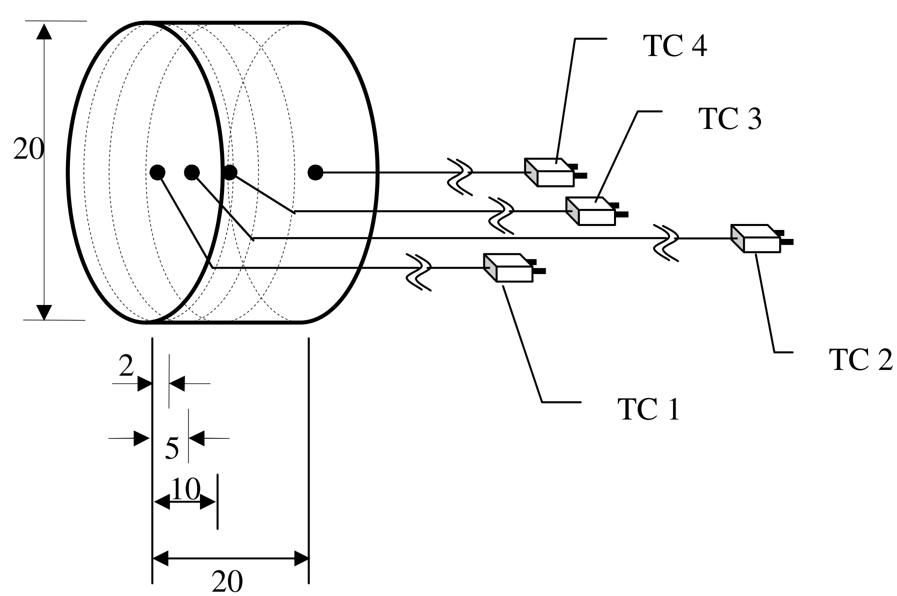


6. RESULTS AND DISCUSSIONS

SAMPLE 1

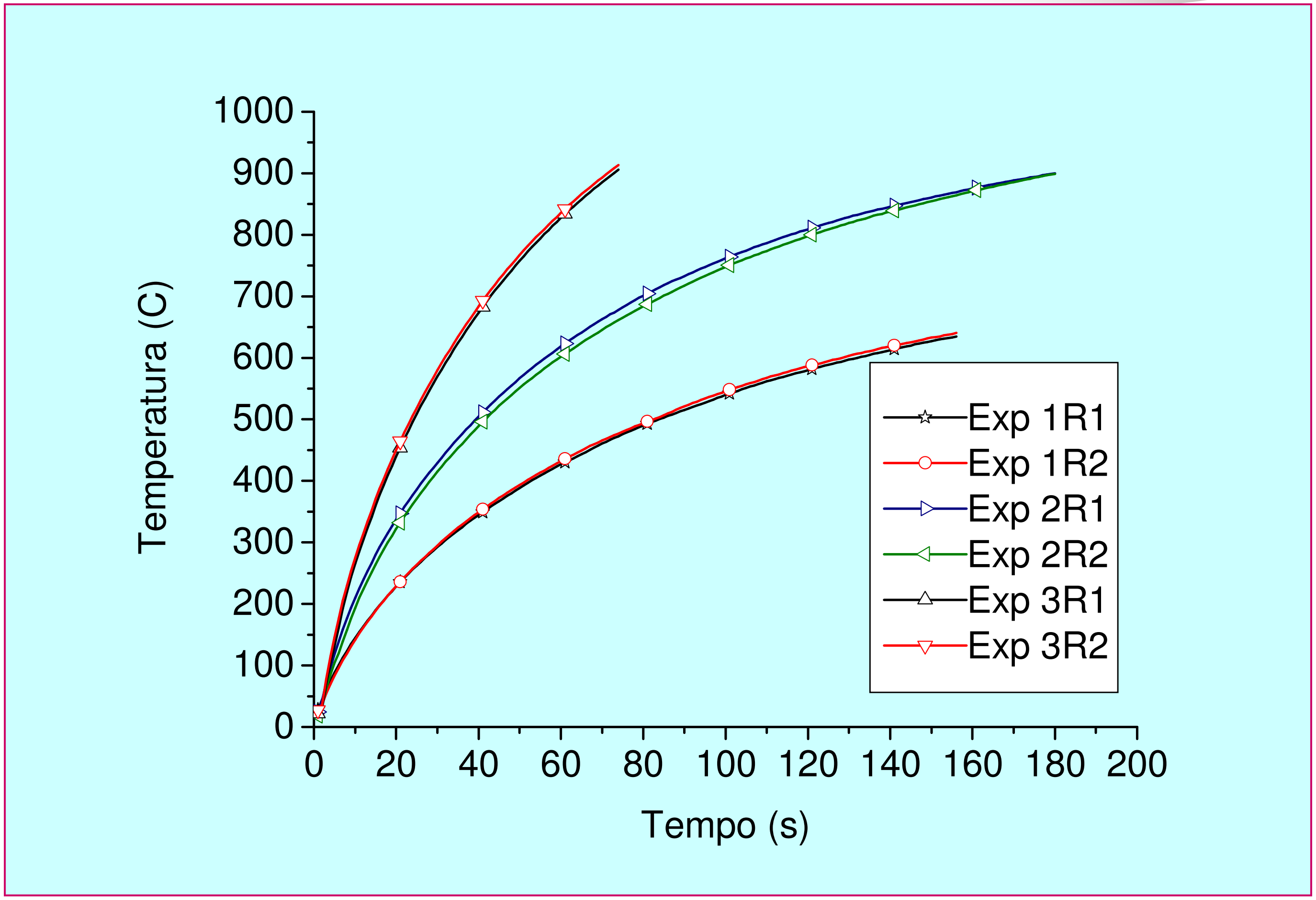


SAMPLE 2



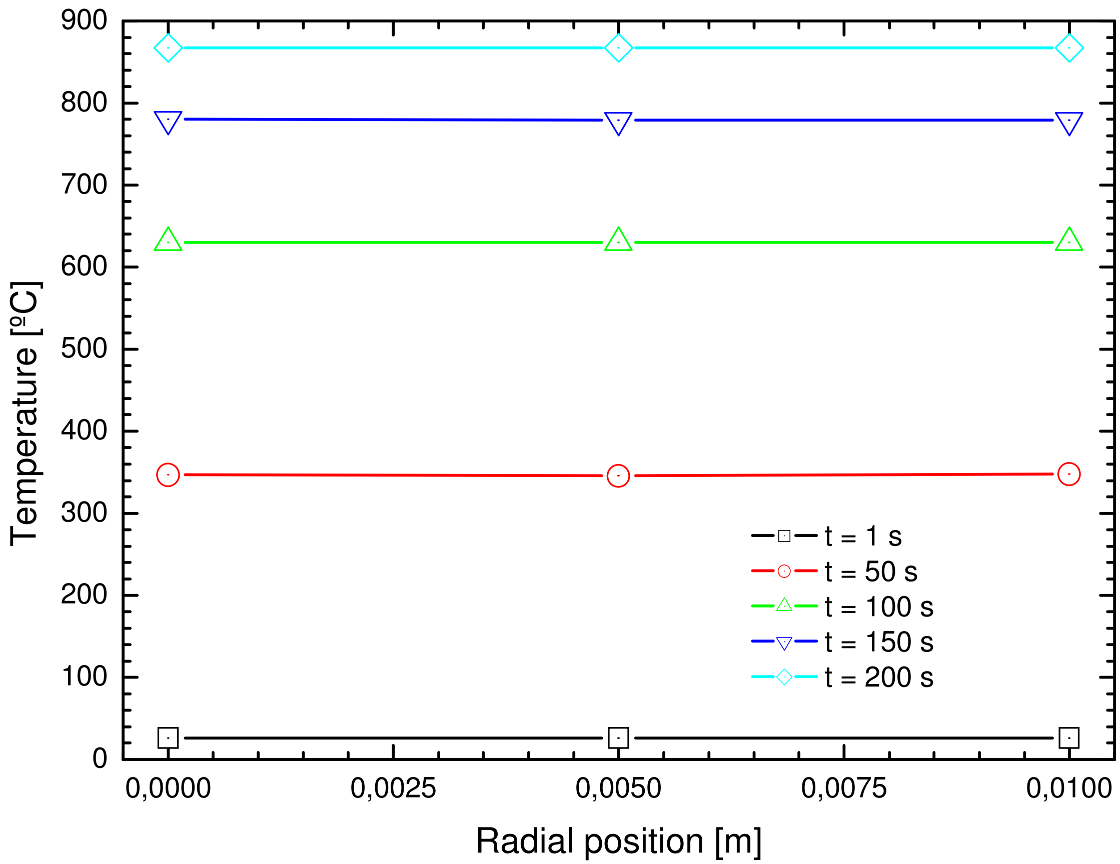
6. RESULTS AND DISCUSSIONS

2 mm below the heated surface



6. RESULTS AND DISCUSSIONS

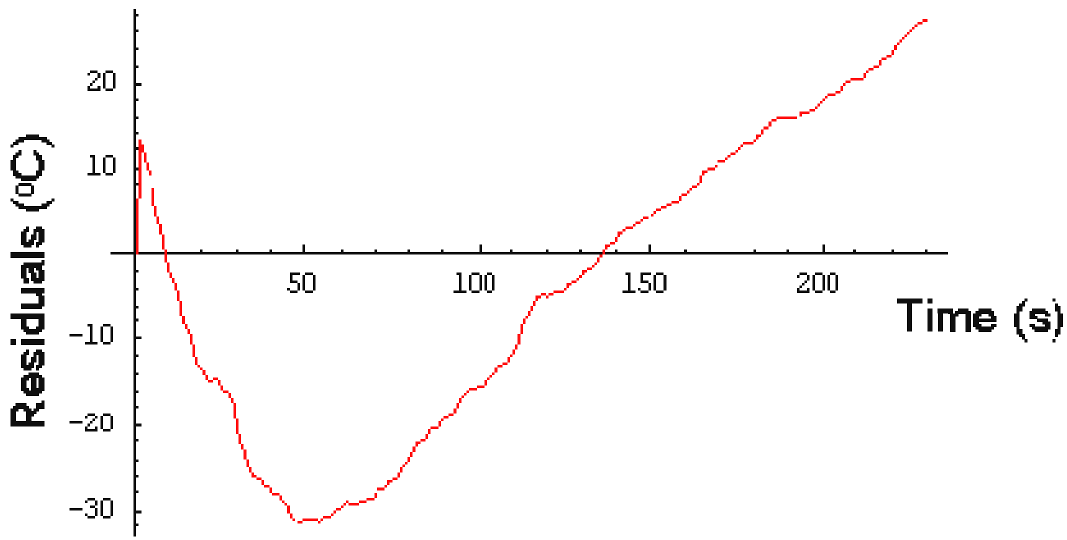
10 mm below the heated surface



6. RESULTS AND DISCUSSIONS

➢ Parameter Estimation

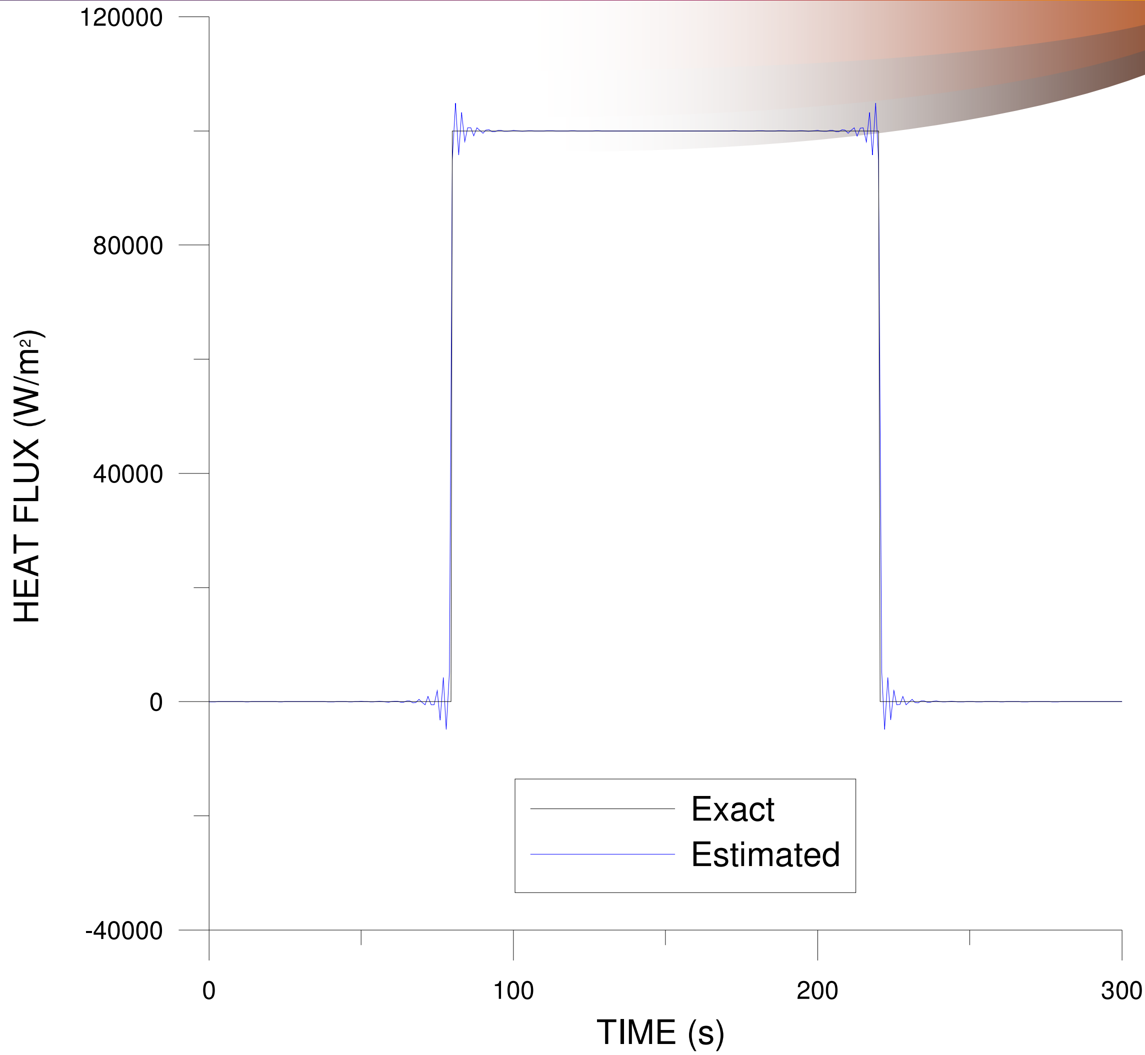
$$Residual = Y(t) - T(t)$$



Large and Correlated Residuals

6. RESULTS AND DISCUSSIONS

- Function Estimation
- Simulated Measurements



6. RESULTS AND DISCUSSIONS

Sample 2 - Distance to the sample = 200 mm
Sensor 3 used as BC – Final Time = 200 s