CONSTITUTIVE PARAMETER ESTIMATION OF SOLIDS

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OUTLINE

- INTRODUCTION
- CONSTITUTIVE EQUATION
- PARAMETER ESTIMATION
- EXPERIMENTAL SETUP
- DIRECT PROBLEM
- RESULTS
- CONCLUDING REMARKS

INTRODUTION

- GOAL
  - ✓ Modelling and identification of viscoelastic materials.

- MOTIVATION
  - ✓ The mechanical model identification for such materials enables more reliable mathematical models towards structures which contains viscoelastic characteristics;
  - ✓ To obtain finite element models for such structures aiming at analyzing their responses to different types of situations and environments.

CONSTITUTIVE EQUATION

- APPROACH
  - ✓ Internal variable based model

    α
- CHARACTERISTICS
  - ✓ Thermodynamic of irreversible processes
  - ✓ Time domain equations
  - ✓ Causal system

CONSTITUTIVE EQUATION

- FREE ENERGY FUNCTION

$$\psi = \psi(\mathbf{F}, T, \boldsymbol{\alpha})$$

- PSEUDO-POTENTIAL OF DISSIPATION

$$\varphi = \varphi(\dot{\mathbf{F}}, \dot{\boldsymbol{\alpha}}, \mathbf{q}/T)$$

- CLAUSIUS-DUHEM CONSTRAINTS

$$\frac{z}{\rho}\mathbf{P}(\mathbf{F}, \dot{\mathbf{F}}, T, \mathbf{g}, \boldsymbol{\alpha}) = \frac{\partial \hat{\psi}}{\partial \mathbf{F}}(\mathbf{F}, T, \boldsymbol{\alpha}) + \frac{\partial \hat{\varphi}}{\partial \dot{\mathbf{F}}}(\dot{\mathbf{F}}, \dot{\boldsymbol{\alpha}}, \mathbf{q}/T)$$

$$\frac{\partial \hat{\varphi}}{\partial \dot{\boldsymbol{\alpha}}}(\dot{\mathbf{F}}, \dot{\boldsymbol{\alpha}}, \mathbf{q}/T) + \frac{\partial \hat{\psi}}{\partial \boldsymbol{\alpha}}(\mathbf{F}, T, \boldsymbol{\alpha}) = 0$$

$$\frac{\partial \hat{\varphi}}{\partial (\mathbf{q}/T)} + \frac{1}{\rho}\mathbf{g} = \mathbf{0}$$

CONSTITUTIVE EQUATION

- HYPOTHESIS
  - ✓ Small deformations
  - ✓ Uniform temperature
  - ✓ Null heat flux

$$\psi = \hat{\psi}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha})$$

$$\varphi = \hat{\varphi}(\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\alpha}})$$

$$\boldsymbol{\varepsilon}(\mathbf{X}, t) = \frac{1}{2}[\nabla \mathbf{u}(\mathbf{X}, t) + \nabla \mathbf{u}^T(\mathbf{X}, t)]$$

CONSTITUTIVE EQUATION

- FREE ENERGY AND DISSIPATION POT.

$$\psi(\varepsilon, \boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^I) = \frac{1}{2\rho}\left[E\,\varepsilon \cdot \varepsilon + \sum_{r=1}^{I} E_r\,(\varepsilon - \boldsymbol{\xi}^r) \cdot (\varepsilon - \boldsymbol{\xi}^r)\right]$$

$$\varphi(\dot{\varepsilon}, \dot{\boldsymbol{\xi}}^1, \ldots, \dot{\boldsymbol{\xi}}^I) = \frac{1}{2\rho}\left[\eta\,\dot{\varepsilon} \cdot \dot{\varepsilon} + \sum_{r=1}^{I} \eta_r\,\dot{\boldsymbol{\xi}}^r \cdot \dot{\boldsymbol{\xi}}^r\right]$$

- CONSTITUTIVE EQUATION

$$\boldsymbol{\sigma} = E\,\varepsilon + \sum_{r=1}^{I} E_r\,(\varepsilon - \boldsymbol{\xi}^r) + \eta\,\dot{\varepsilon}$$

$$\dot{\boldsymbol{\xi}}^r = b_r\,(\varepsilon - \boldsymbol{\xi}^r), \quad r = 1, \ldots, I \qquad b_r = \frac{E_r}{\eta_r}$$

PARAMETER ESTIMATION

- UNKNOWN PARAMETER VECTOR

$$\mathbf{p} = \{p_1, p_2, \ldots, p_{N_p}\}^T$$

- OBJECTIVE FUNCTION

$$S(\mathbf{p}) = [\mathbf{Y} - \bar{\mathbf{Y}}]^T [\mathbf{Y} - \bar{\mathbf{Y}}]$$

- INVERSE PROBLEM

$$\min_{\mathbf{p}} S(\mathbf{p})$$

PARAMETER ESTIMATION

- CONSTRAINTS
  - ✓ MATHEMATICALLY

    $$c_i(\mathbf{p}) \geq 0, \qquad i = 1, \ldots, q$$

  - ✓ PHYSICALLY

    $E > 0 \quad E_k > 0 \quad b_k > 0$

- CONSTRAINED PROBLEM

$$\min_{\mathbf{p}} S(\mathbf{p}) \qquad \mathbf{p} \in \mathcal{P}$$

PARAMETER ESTIMATION

- FROM A CONSTRAINED TO AN UNCONSTRAINED PROBLEM
  - ✓ Interior penalty function method
  - ✓ Extended objective function

$$S_e(\mathbf{p}) = S(\mathbf{p}) + \sum_{i=1}^{q} \zeta_i(\mathbf{p})$$

  - ✓ Barrier functions

$$\zeta_i(\mathbf{p}) = \frac{z_i}{c_i(\mathbf{p})}$$

PARAMETER ESTIMATION

- MODIFIED LEVENBERG MARQUARDT PARAMETER ESTIMATION TECHNIQUE
  - ✓ Sensitivity matrix J

$$J_{ij} = \frac{\partial Y_i}{\partial p_j}, \qquad i = 1, \ldots, N_s \times N_f \quad \text{and} \quad j = 1, \ldots, N_p$$

  - ✓ Parameter updating

$$[\mathbf{J}^{(k)^T}\mathbf{J}^{(k)} + \mu^{(k)}\boldsymbol{\Gamma}^{(k)} + \mathbf{H}^{(k)}]\Delta\mathbf{p}^{(k)} = -\mathbf{J}^{(k)^T}[\mathbf{Y}^{(k)} - \bar{\mathbf{Y}}] + \mathbf{g}^{(k)}$$

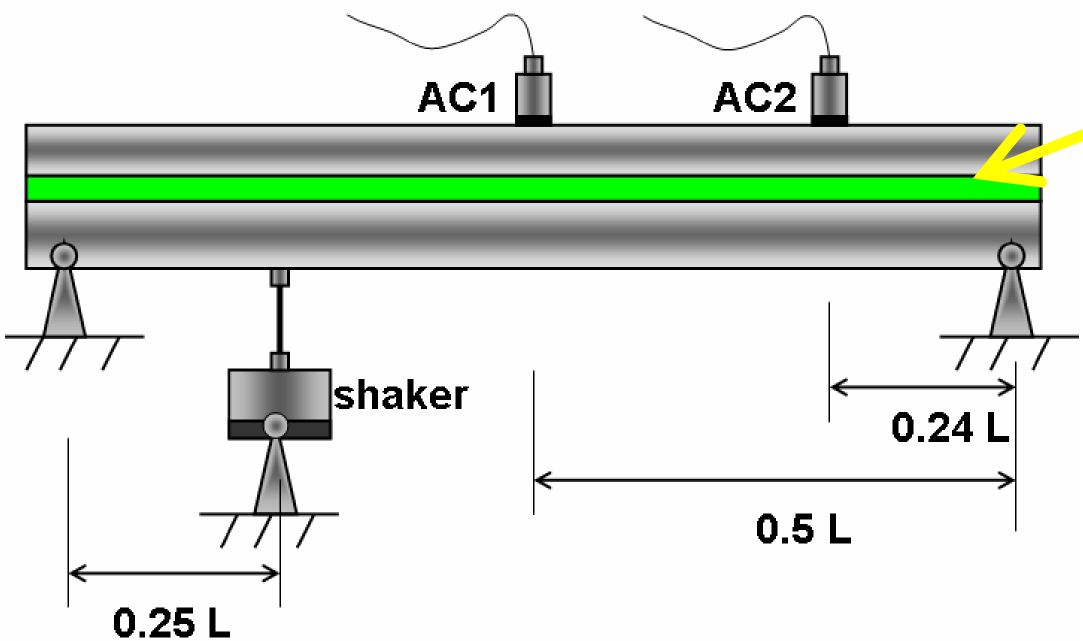
  - ✓ Vector **g** and matrix **H**

$$g_j = -\sum_{r=1}^{q} \frac{\partial \zeta_r}{\partial p_j} \qquad H_{i,j} = \sum_{r=1}^{q} \frac{\partial^2 \zeta_r}{\partial p_i \partial p_j}, \qquad i, j = 1, \ldots, N_p$$

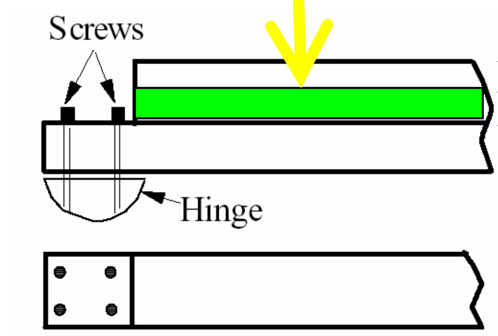
EXPERIMENTAL SETUP

- SKETCH

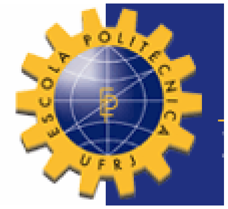
VHB 4950 – 3M

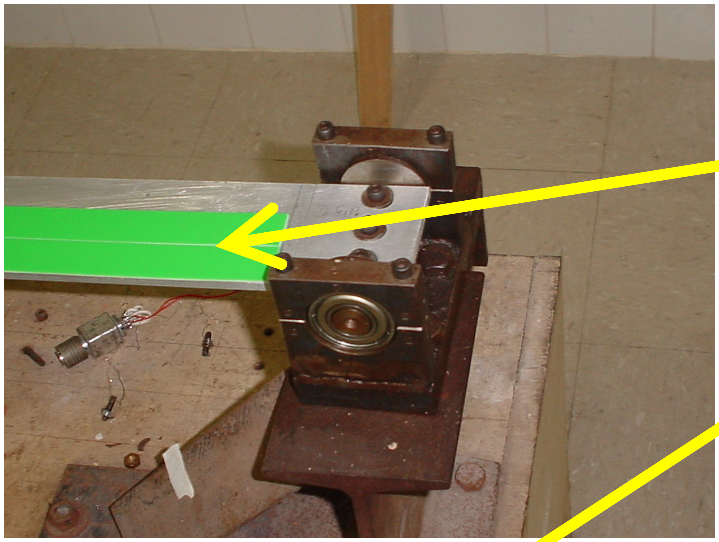


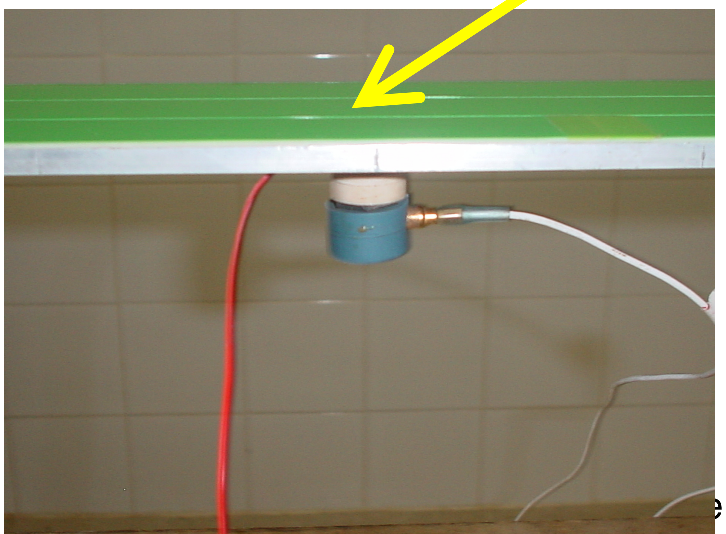
Detail of the end constraints

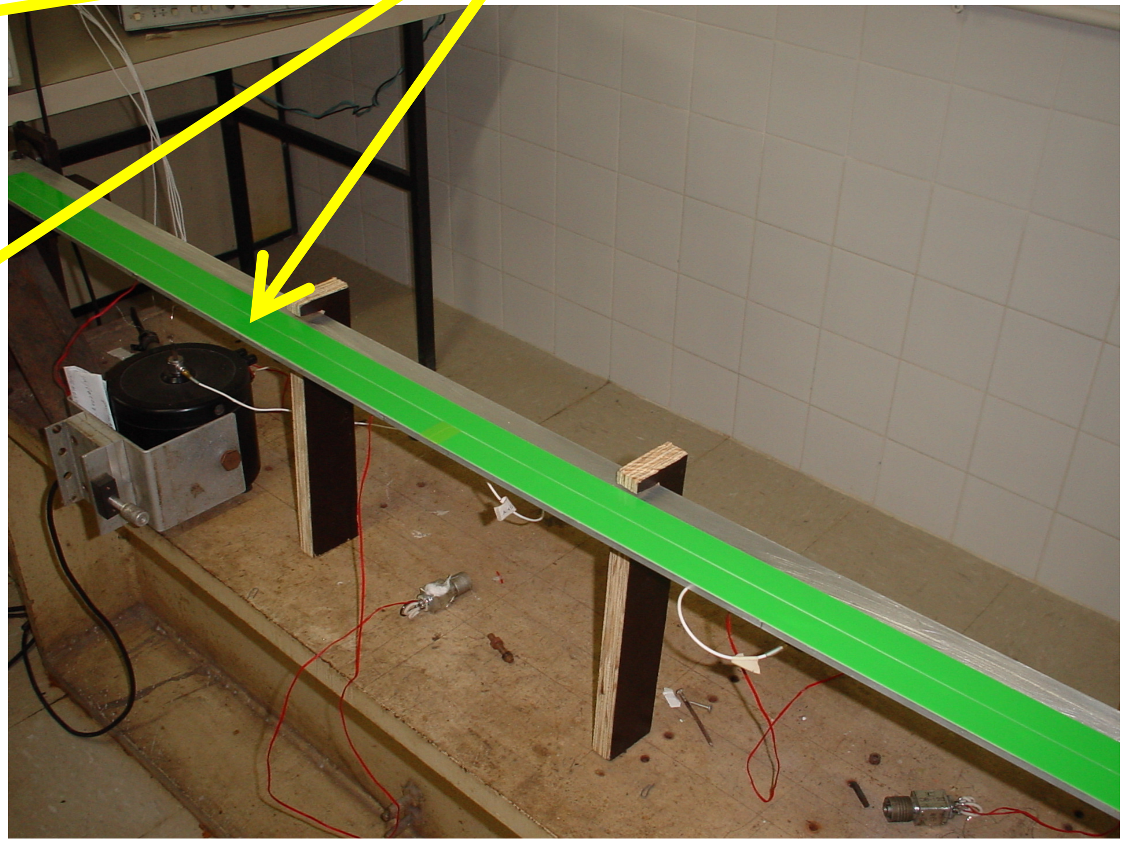


EXPERIMENTAL SETUP



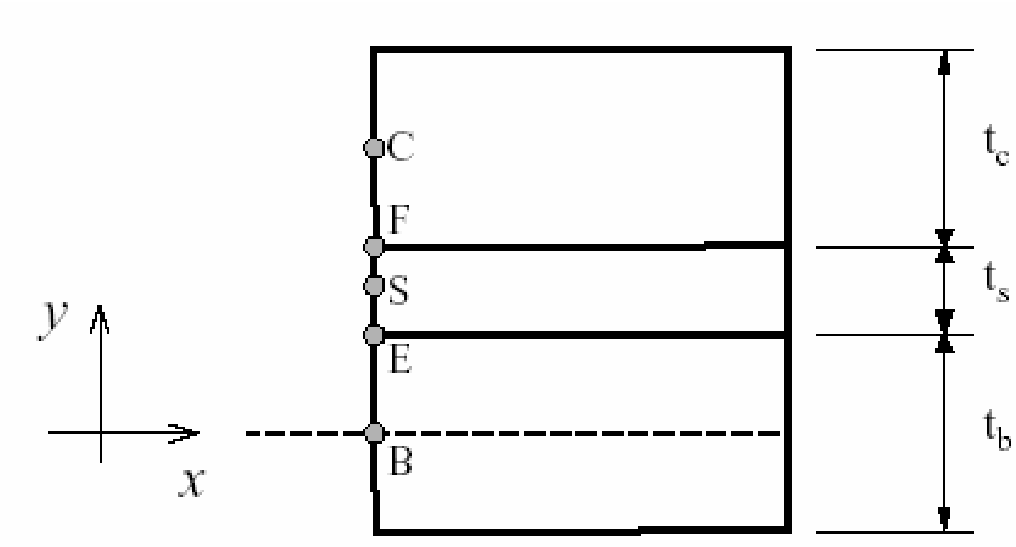


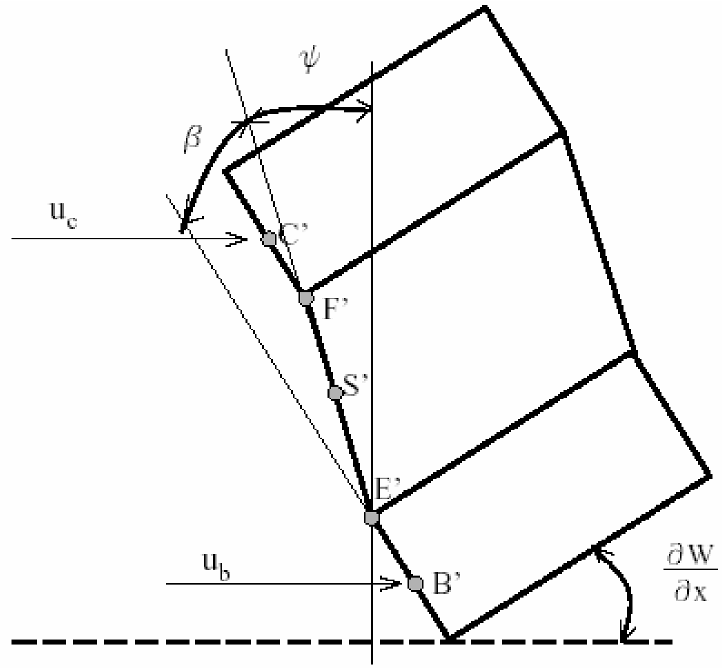




DIRECT PROBLEM

- FINITE ELEMENT MODEL





DIRECT PROBLEM

- CONSTITUTIVE EQUATION

$$\begin{Bmatrix} \sigma_x^{(2)} \\ \sigma_y^{(2)} \\ \sigma_{xy}^{(2)} \end{Bmatrix} = \left[\begin{pmatrix} G_1 & G_{12} & 0 \\ G_{21} & G_2 & 0 \\ 0 & 0 & G_3 \end{pmatrix} + \sum_{p=1}^{I} \begin{pmatrix} G_1^{(p)} & 0 & 0 \\ 0 & G_2^{(p)} & 0 \\ 0 & 0 & G_3^{(p)} \end{pmatrix} \right] \begin{Bmatrix} \varepsilon_x^{(2)} \\ \varepsilon_y^{(2)} \\ \varepsilon_{xy}^{(2)} \end{Bmatrix} + (-1) \sum_{p=1}^{I} \begin{pmatrix} G_1^{(p)} & 0 & 0 \\ 0 & G_2^{(p)} & 0 \\ 0 & 0 & G_3^{(p)} \end{pmatrix} \begin{Bmatrix} \xi_x^{(p)} \\ \xi_y^{(p)} \\ \xi_{xy}^{(p)} \end{Bmatrix}$$

$$\begin{pmatrix} \eta_1^{(p)} & 0 & 0 \\ & \eta_2^{(p)} & 0 \\ 0 & 0 & \eta_3^{(p)} \end{pmatrix} \begin{Bmatrix} \dot{\xi}_x^{(p)} \\ \dot{\xi}_y^{(p)} \\ \dot{\xi}_{xy}^{(p)} \end{Bmatrix} + \begin{pmatrix} G_1^{(p)} & 0 & 0 \\ 0 & G_2^{(p)} & 0 \\ 0 & 0 & G_3^{(p)} \end{pmatrix} \begin{Bmatrix} \xi_x^{(p)} \\ \xi_y^{(p)} \\ \xi_{xy}^{(p)} \end{Bmatrix} = \begin{pmatrix} G_1^{(p)} & 0 & 0 \\ 0 & G_2^{(p)} & 0 \\ 0 & 0 & G_3^{(p)} \end{pmatrix} \begin{Bmatrix} \varepsilon_x^{(2)} \\ \varepsilon_y^{(2)} \\ \varepsilon_{xy}^{(2)} \end{Bmatrix} \qquad p = 1, \ldots, I$$

DIRECT PROBLEM

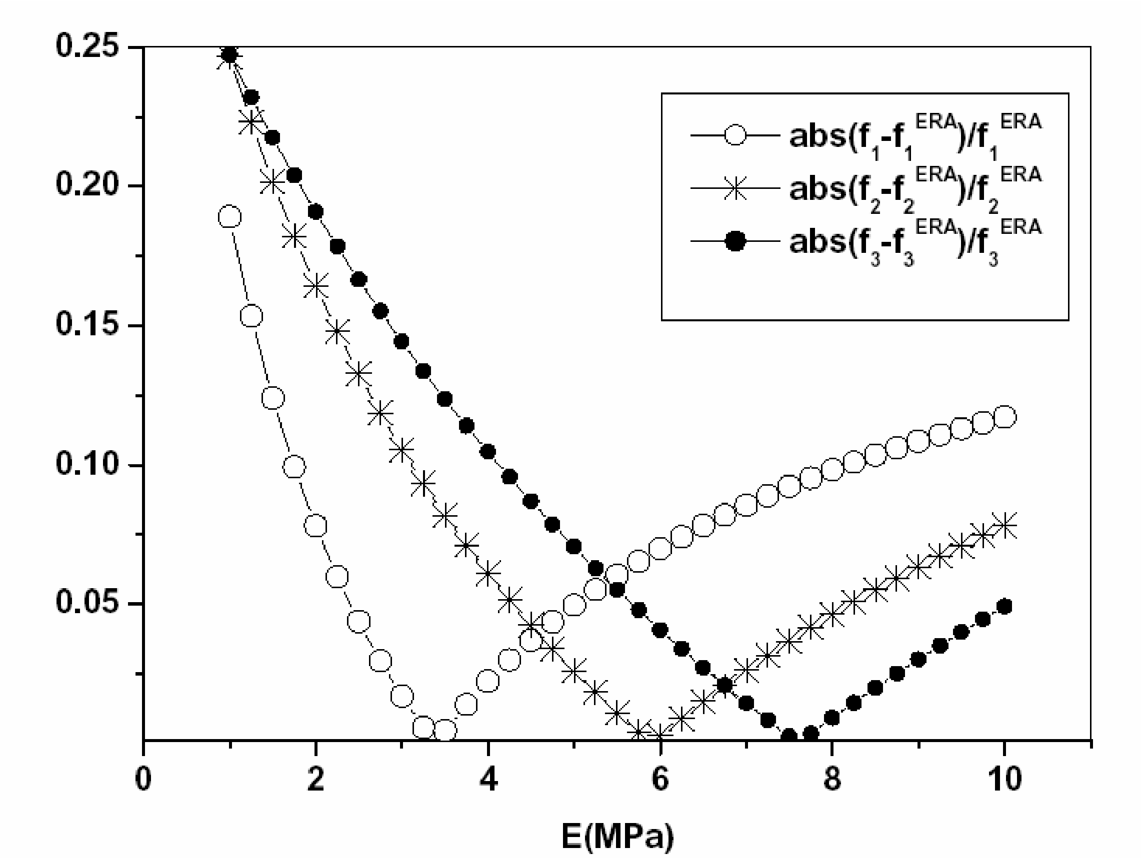
- CONSTITUTIVE EQUATION
  - ✓ Hypothesis: dissipation within the viscoelastic tape is basically due to shearing deformation
  - ✓ Therefore, for the shearing within the viscoelastic layer:

$$\boldsymbol{\sigma} = E\,\boldsymbol{\varepsilon} + \sum_{r=1}^{I} E_r\,(\boldsymbol{\varepsilon} - \boldsymbol{\xi}^r) + \eta\,\dot{\boldsymbol{\varepsilon}}$$

$$\dot{\boldsymbol{\xi}}^r = b_r\,(\boldsymbol{\varepsilon} - \boldsymbol{\xi}^r), \quad r = 1, \ldots, I \qquad b_r = \frac{E_r}{\eta_r}$$

RESULTS (NI=1)

- INITIAL GUESS
  - ✓ Range for E: Comparison of the first 3 FRF peaks



1 < E < 10 (MPa)

RESULTS (NI=1)

- PARAMETERIZATION

$E = p_1 \times 10^6$ $\quad E_1 = p_2 \times 10^6$ $\quad b_1 = p_3$

- EXPERIMENTAL DATA
  - ✓ FRFs of the first and second accelerometer from 2.25 Hz to 99.75 Hz (391 points each)
- OBJECTIVE FUNCTION
  - ✓ Difference between the experimental FRFs and the ones obtained by the model
- INITIAL GUESS: $p_1 = 5$, $p_2 = p_3 = 0.1$

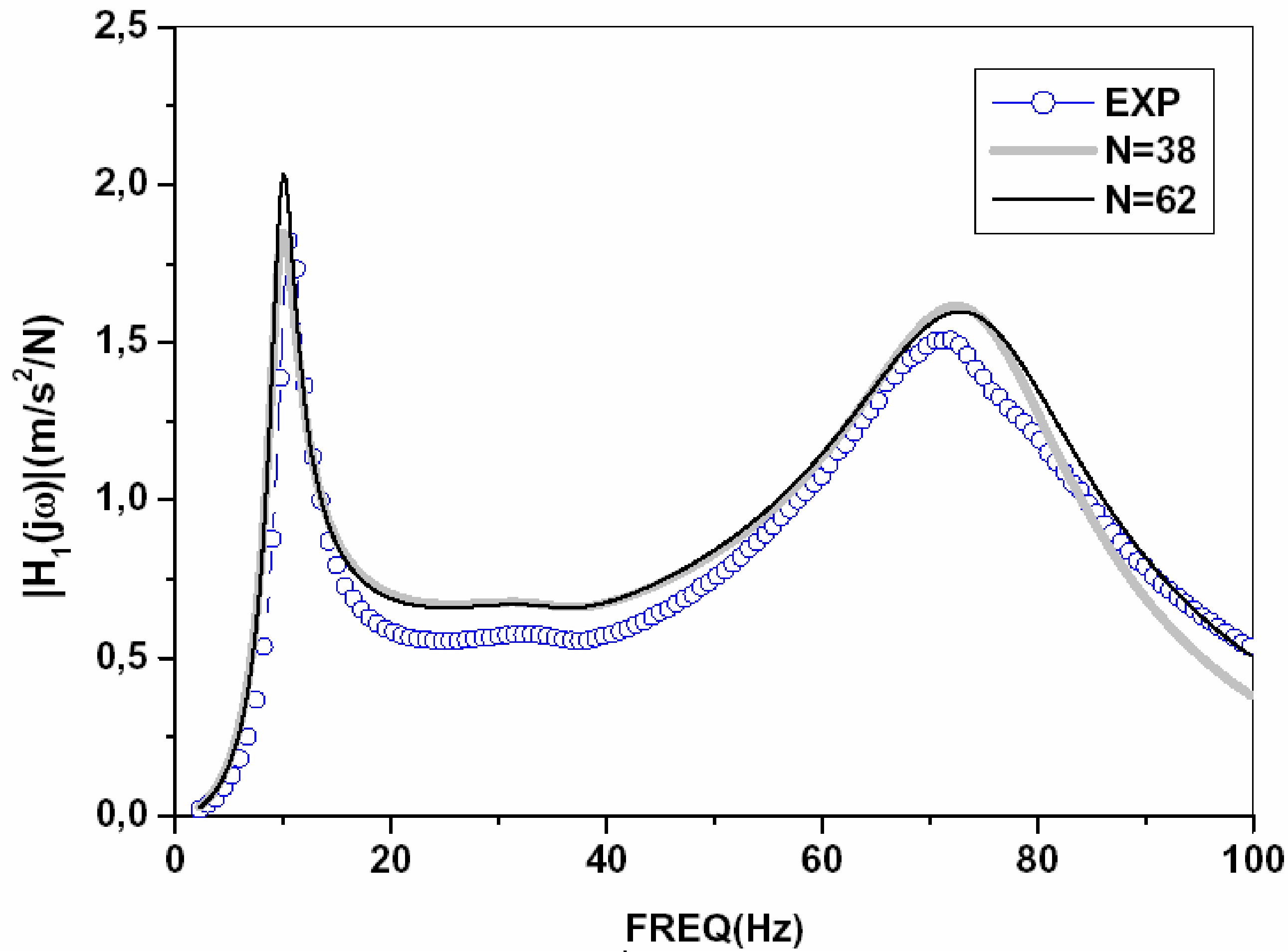
RESULTS (NI=1)

- ESTIMATED PARAMETERS
  - ✓ Different mesh sizes (N)

	p_1	p_2	p_3
N =38	1.403	10.558	537.42
N =42	1.516	10.943	567.47
N =54	1.647	12.353	662.61
N =62	1.628	12.385	656.22

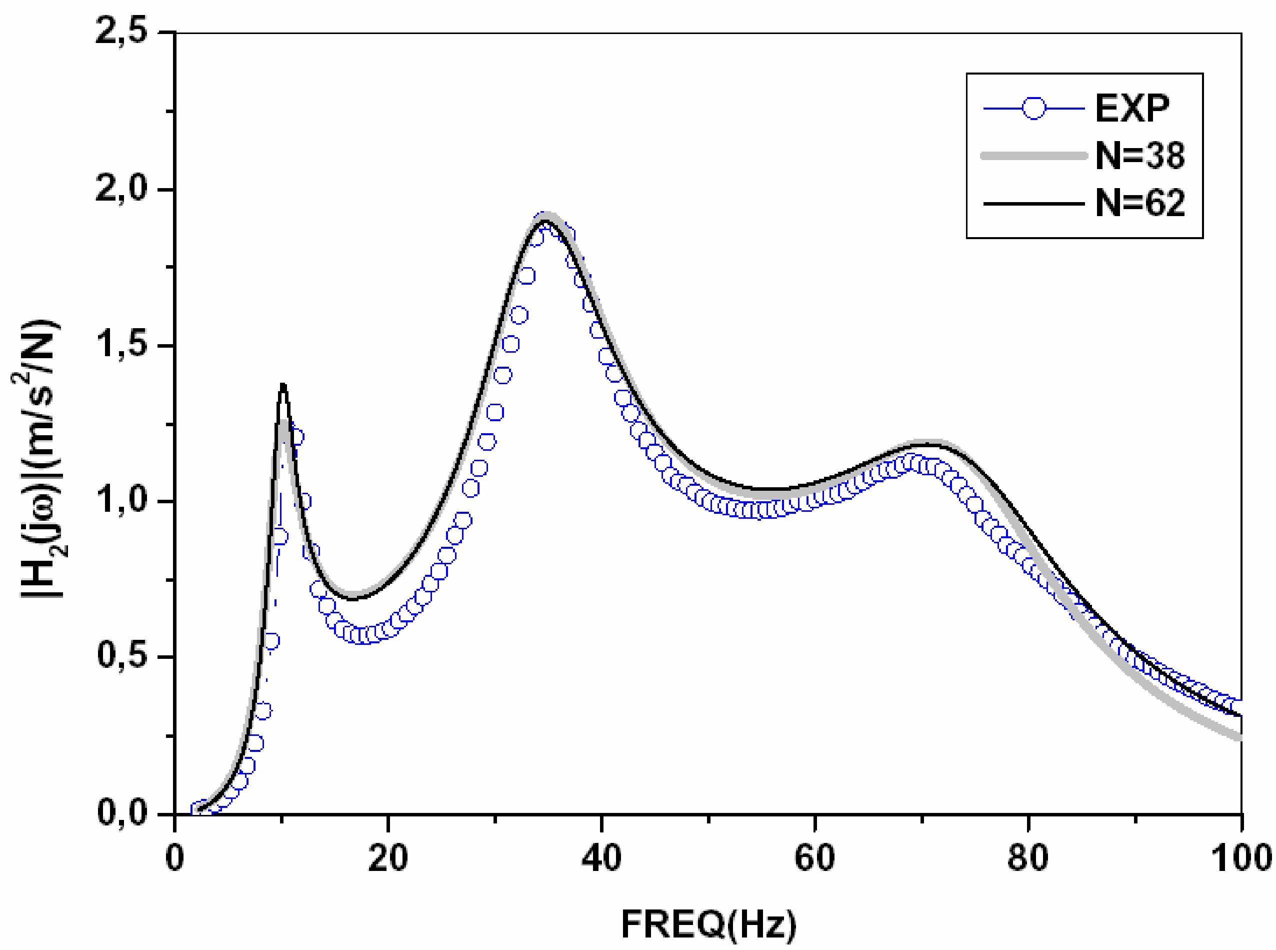
RESULTS (NI=1)

- FIRST ACCELEROMETER



RESULTS (NI=1)

- SECOND ACCELEROMETER



RESULTS (NI=1)

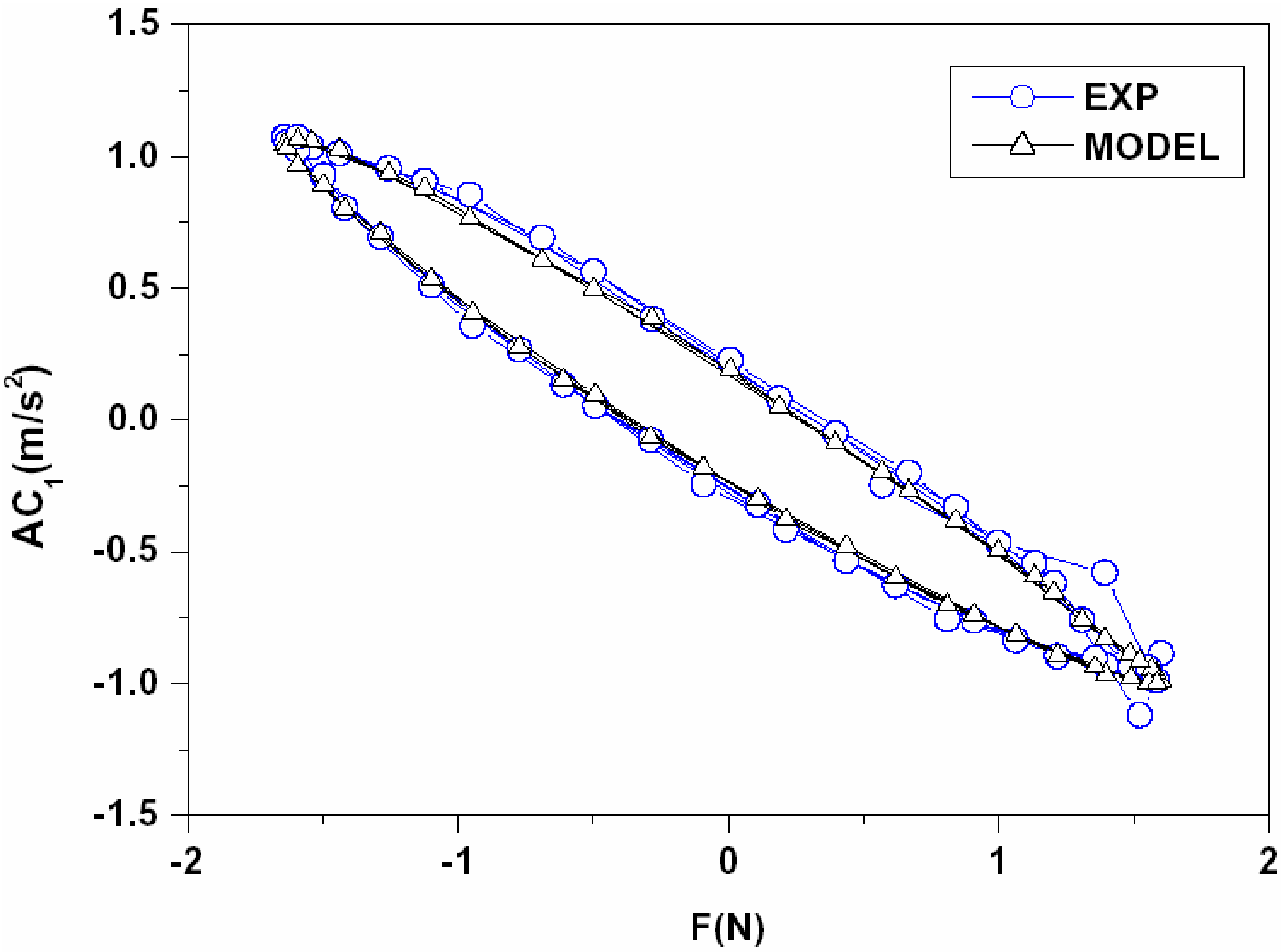
- NORMS H_2 AND H_∞

$\frac{\|\Delta H(j\omega)\|_2}{\|H(j\omega)\|_2}$	$AC1$	$AC2$
N =38	0.2327	0.1764
N =42	0.2286	0.1771
N =54	0.2236	0.1803
N =62	0.2227	0.1791

$\frac{\|\Delta H(j\omega)\|_\infty}{\|H(j\omega)\|_\infty}$	$AC1$	$AC2$
N =38	0.6432	0.4274
N =42	0.6393	0.4267
N =54	0.6505	0.4317
N =62	0.6402	0.4256

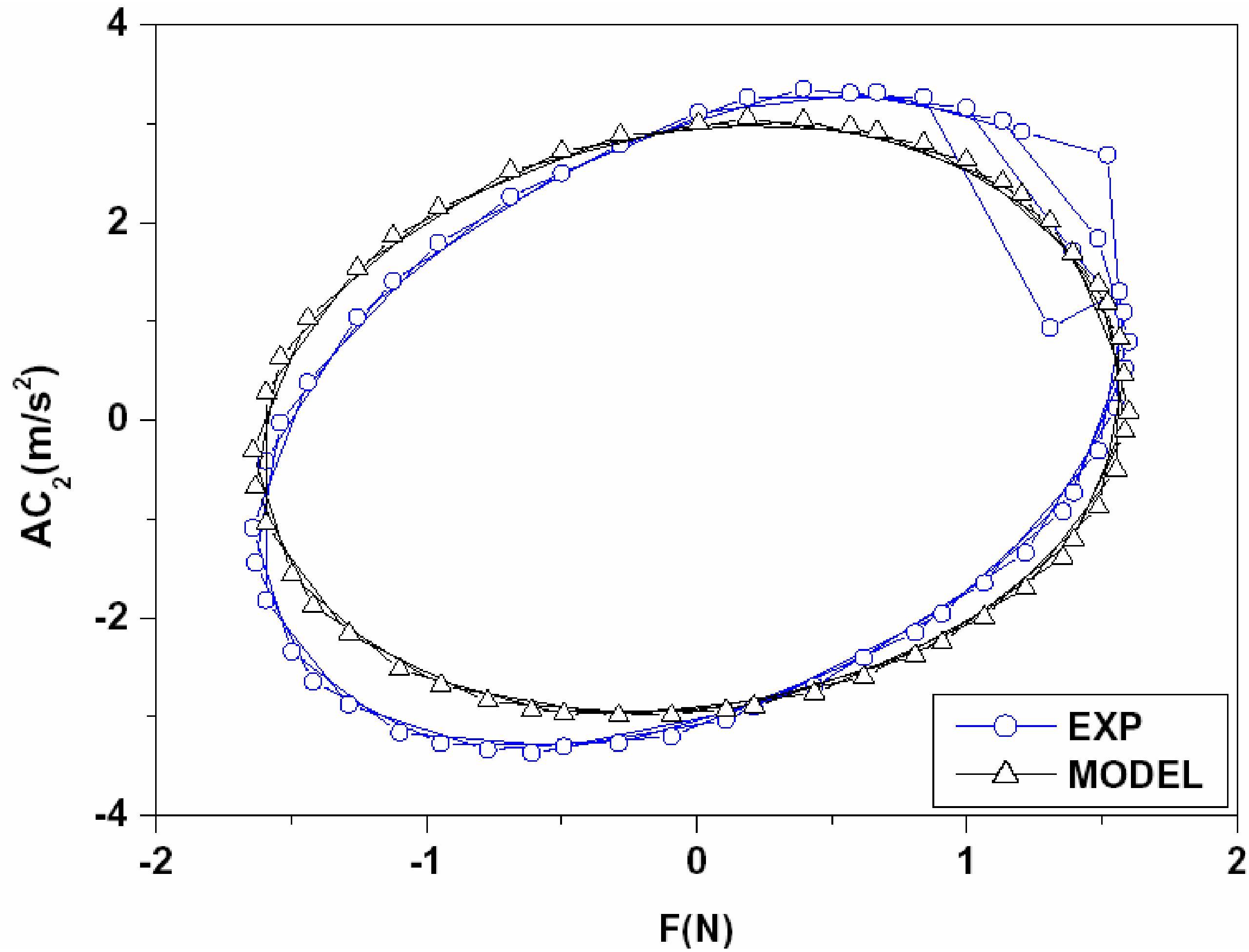
RESULTS (NI=1)

- TIME DOMAIN VALIDATION: 35 Hz sine



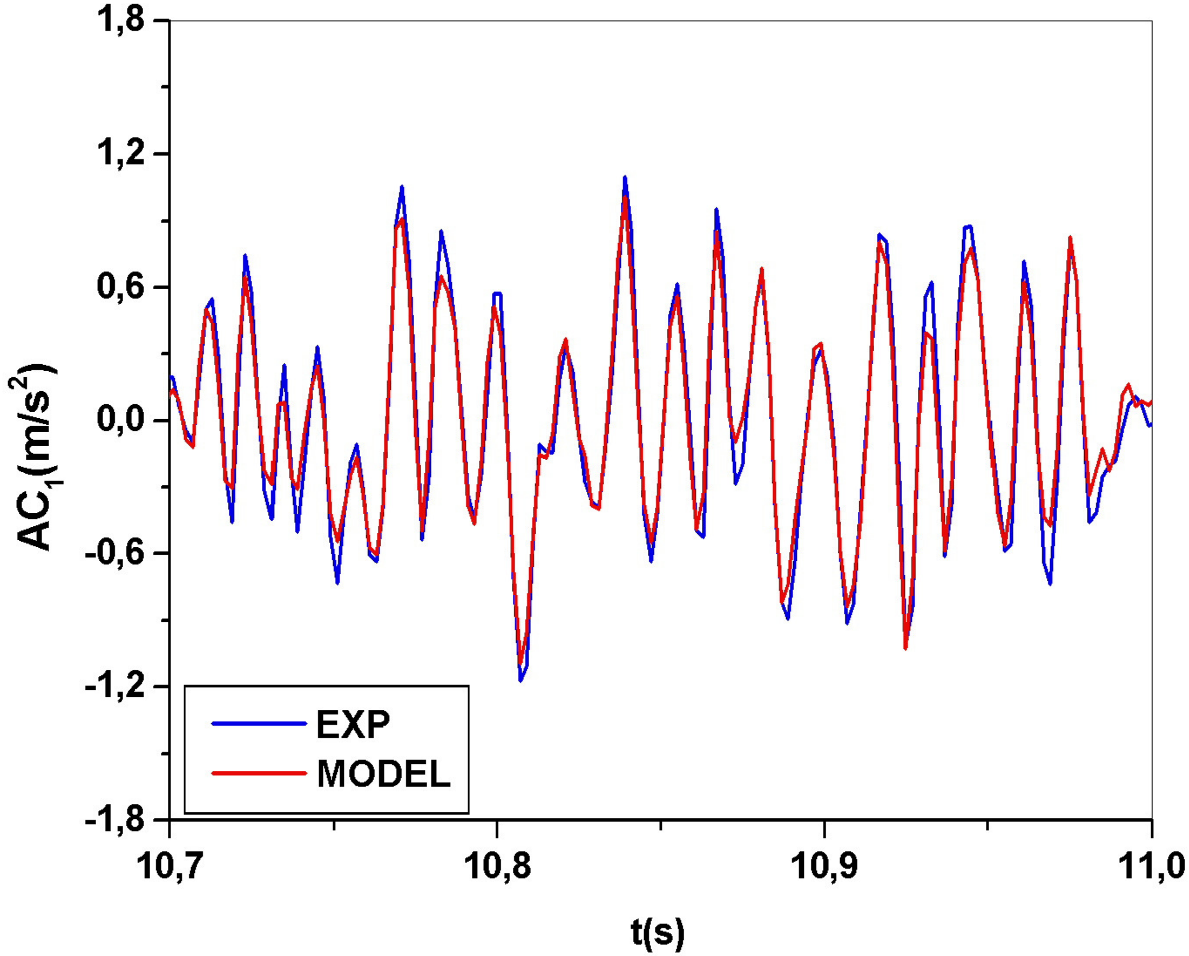
RESULTS (NI=1)

- TIME DOMAIN VALIDATION: 35 Hz sine



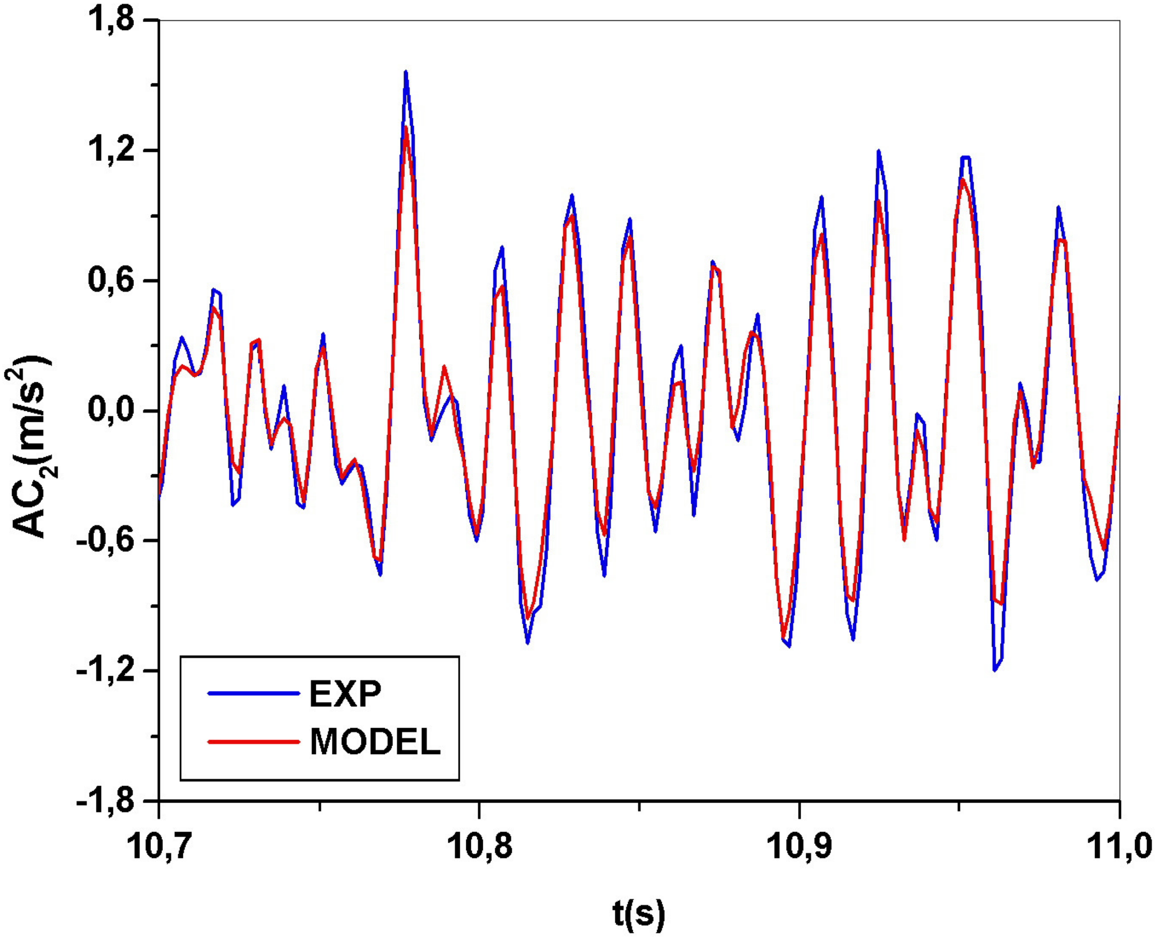
RESULTS (NI=1)

- TIME DOMAIN VALIDATION: WHITE NOISE



RESULTS (NI=1)

- TIME DOMAIN VALIDATION: WHITE NOISE

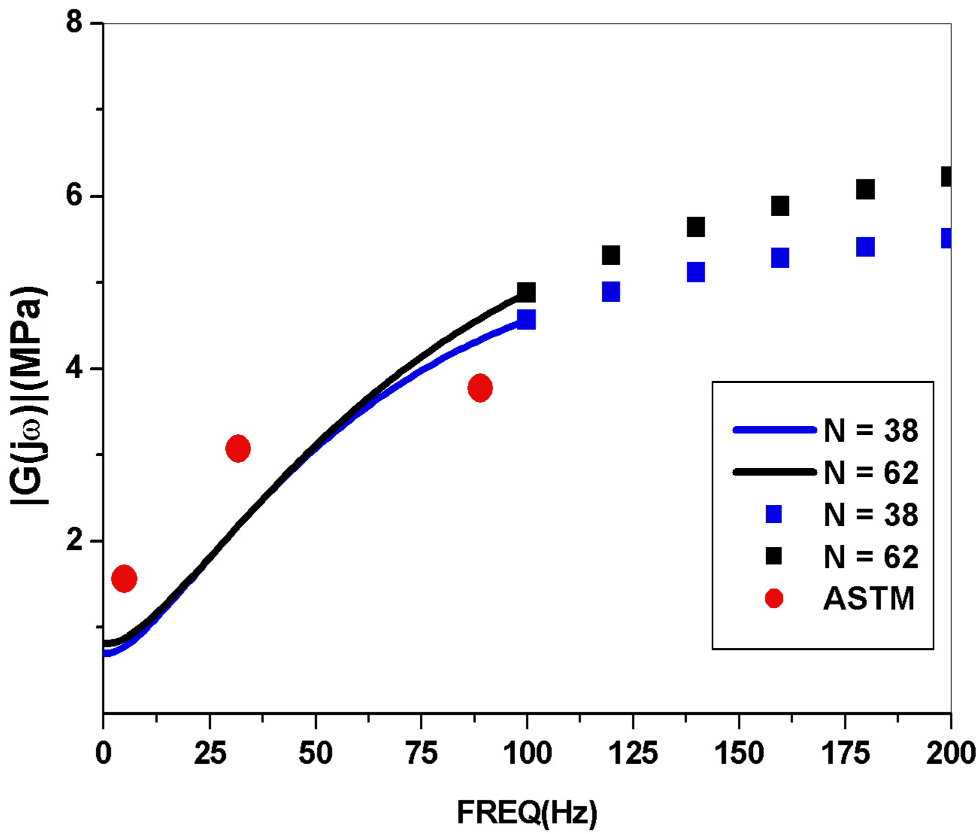


RESULTS (NI=1)

- ASTM VALIDATION
  - ✓Faisca *et al.* determined the complex shear modulus of a VHB tape identical to this one except for the thickness by means of the well known ASTM method.
  - ✓Comparison of results obtained by Faisca *et al.* and the ones obtained in the present work

RESULTS (NI=1)

- ASTM VALIDATION



Complex shear modulus: O: ASTM, continuous line: estimated, squares: extrapolation

RESULTS (NI=2)

- PARAMETERIZATION

$E = p_1 \times 10^6 \quad E_1 = p_2 \times 10^6 \quad b_1 = p_3$

$E_2 = p_4 \times 10^6 \quad b_2 = p_5$

- REGULARIZATION
  - ✓ Based on last estimation result
  - ✓ Based on the ASTM estimation (Faisca *et al.*)

- OBJECTIVE FUCTION
  - ✓ $S_R(\mathbf{p}) = S_e(\mathbf{p}) + \alpha \cdot f_\alpha(\mathbf{p})$

RESULTS (NI=2)

- REGULARIZATION 1
  - ✓ Rationale: As the last estimation provided results close to reality, one may expect that some parameters will not have large variations for a 2 I.V. based model

$$f_\alpha(\mathbf{p}) = \alpha_p(\tilde{\mathbf{p}} - \mathbf{B}\mathbf{p})^T \mathbf{W}(\tilde{\mathbf{p}} - \mathbf{B}\mathbf{p})$$

$\tilde{\mathbf{p}}$: vector containing the result of the last estimation

$\alpha_p \in \mathbf{R}^+$: regularizing parameter

$$W_{ij} = \begin{cases} \tilde{p}_i^{-2}\delta_{ij}, & \text{if} \quad i = 1,2,3 \\ 0, & otherwise \end{cases} \qquad B_{ij} = \begin{cases} \delta_{ij}, & \text{if} \quad i = 1,2,3 \\ 0, & otherwise \end{cases}$$

RESULTS (NI=2)

- REGULARIZATION 1
  - ✓Several values of α_p from 0.5 to 1.5 have been tested
  - ✓Initial guesses: $p_1 = 1.647$, $p_2 = 12.353$, $p_3 = 662.6$, $p_4 = 10^{-2}$ and $p_5 = 1.0$
  - ✓Best correlation for $\alpha_p = 1.0$ and result:

    $p_1 = 0.1546$, $p_2 = 12.353$, $p_3 = 791.61$,

    $p_4 = 2.12$ and $p_5 = 38.98$

RESULTS (NI=2)

- REGULARIZATION 2
  - ✓ Based on a priori information concerning the complex shear modulus obtained by Faisca *et al.* by means of the ASTM method.

$$f_\alpha(\mathbf{p}) = \alpha_{ASTM}\mathbf{q}^T\mathbf{q}$$

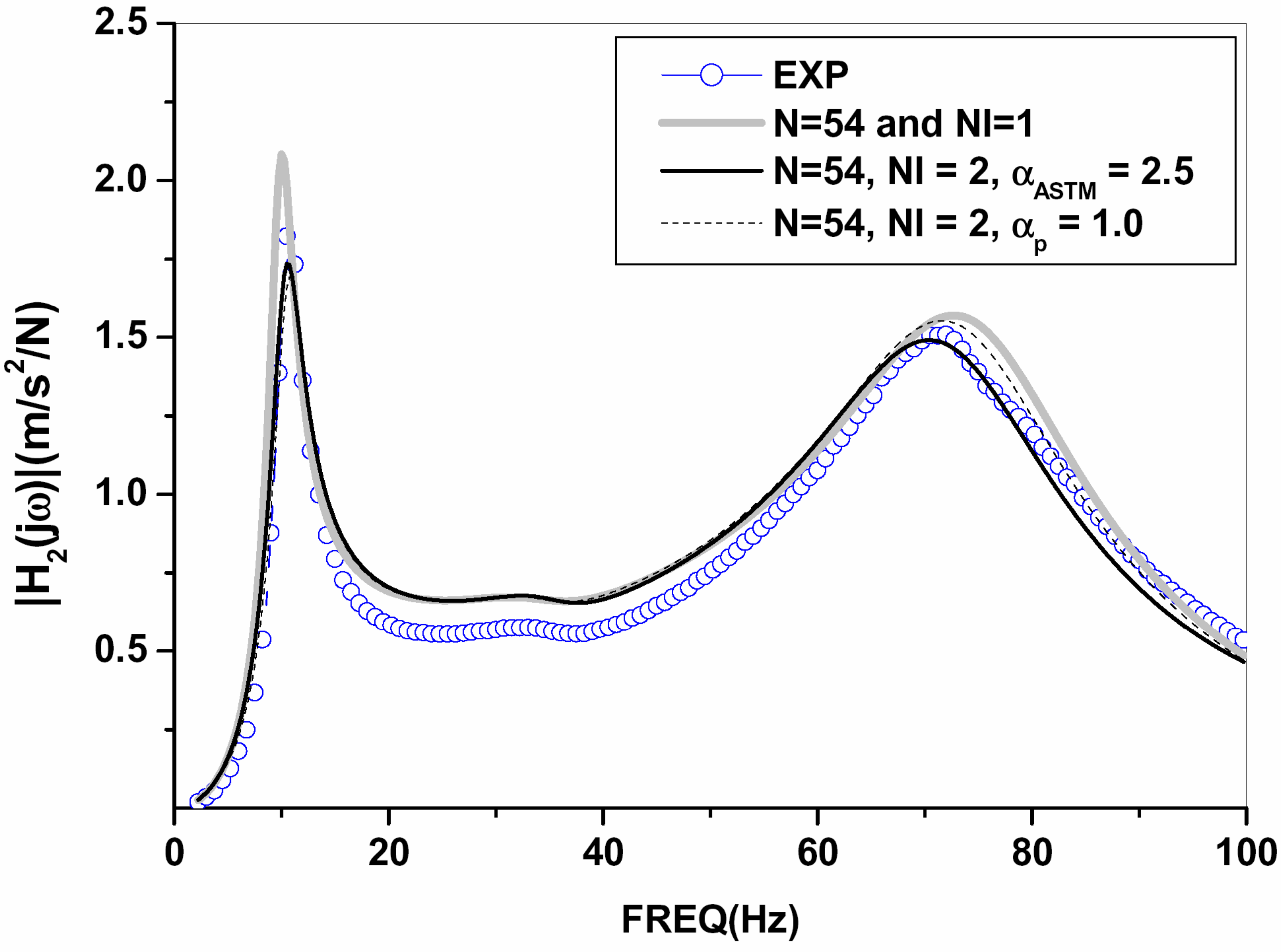
$$q_i = |G_{ASTM}(\omega_i)| - |G(\omega_i, \mathbf{p})|$$

RESULTS (NI=2)

- REGULARIZATION 2
  - ✓Several values of ap from 1 to 10 have been tested
  - ✓Initial guesses: $p_1 = 1$, $p_2 = 1$, $p_3 = 10$, $p_4 = 1$ and $p_5 = 10$
  - ✓Best correlation for $\alpha_{ASTM} = 2.5$ and result: $p_1 = 1.4$, $p_2 = 41.67$, $p_3 = 4662$, $p_4 = 3.63$ and $p_5 = 1999.4$

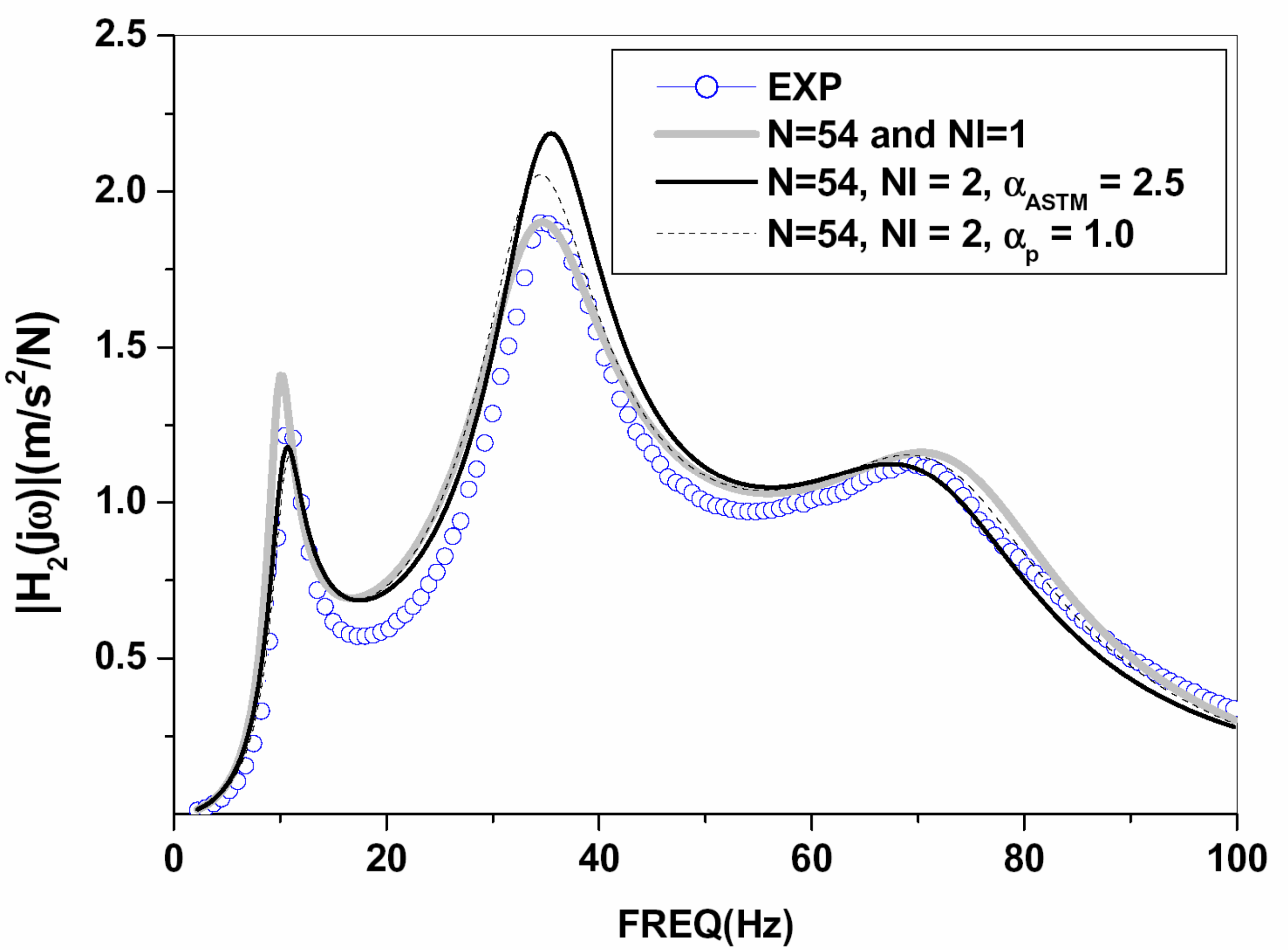
RESULTS (NI=2)

- FIRST ACCELEROMETER



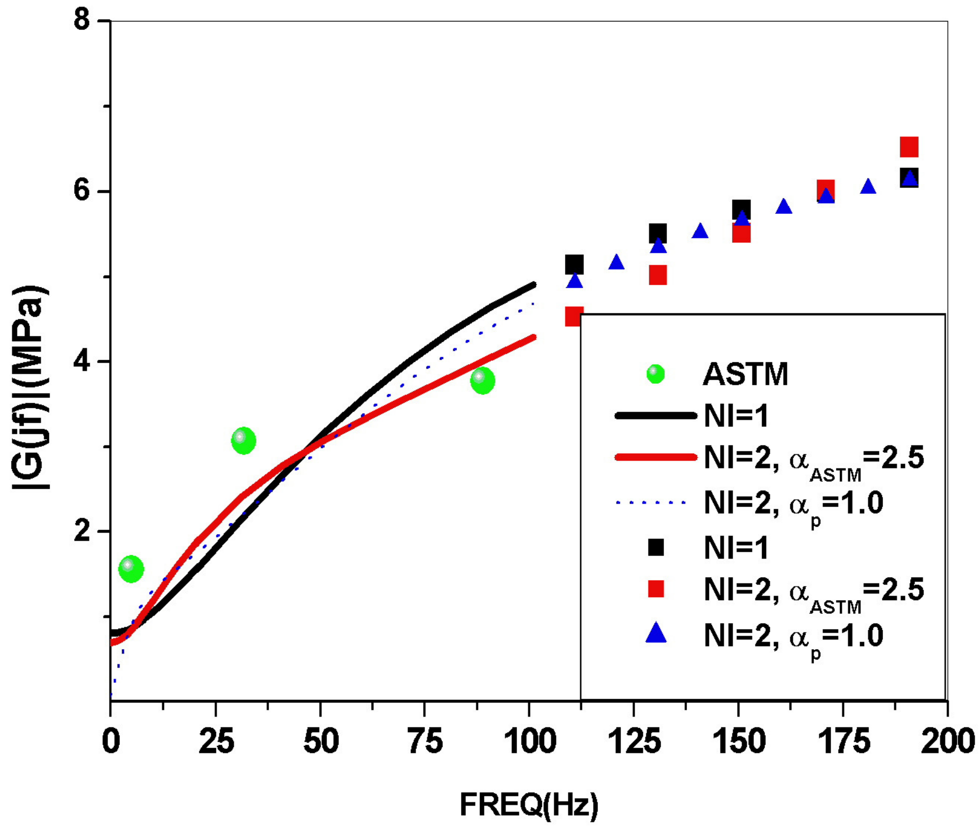
RESULTS (NI=2)

- SECOND ACCELEROMETER



RESULTS (NI=2)

- ASTM VALIDATION



Complex shear modulus: O: ASTM, continuous and dashed lines: estimated, squares and triangles: extrapolation

CONCLUDING REMARKS

- Internal variable based viscoelastic model.
- Modified Levenberg-Marquardt.

- Modeling and estimation assessed on a viscoelastic tape (VHB)

- Effective results encompassing models with one and with two internal variables

- Difficulties
  - ✓ Initial guesses
  - ✓ Regularization required for two internal variable based model

PARAMETER ESTIMATION BASED ON IMAGE CORRELATION DATA

- Digital image correlation is used to obtain the displacement field of a subdomain of a body.
- Inverse problem formulation.
- Parameter estimation

DIGITAL IMAGE CORRELATION (DIC) METHOD

- Optical-numerical full-field surface displacement measuring technique.
- Based on a comparison between two images of the specimen coated by a random speckle pattern in the underformed and in the deformed state.
- The basic principle of DIC method is to search for the maximum correlation between small zones in the underformed and deformed images.

DIGITAL IMAGE CORRELATION (DIC) METHOD

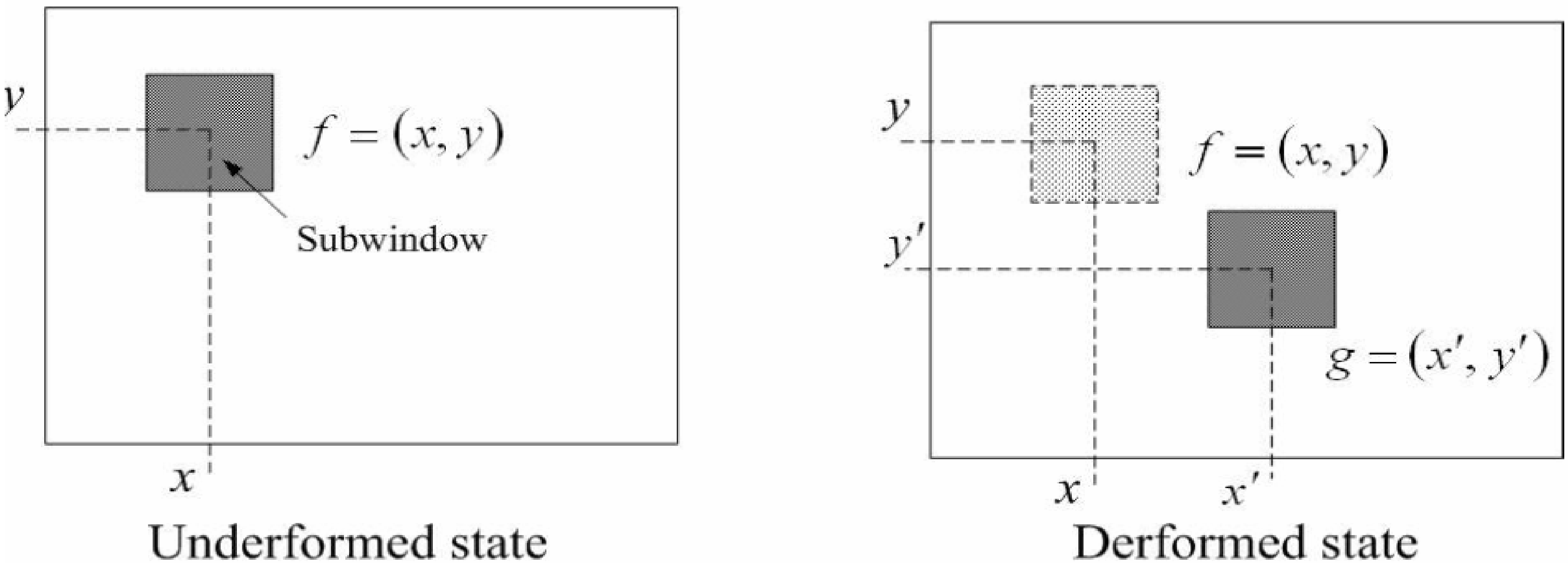


Figure 1. Schematic diagram of the deformation relation.

DIGITAL IMAGE CORRELATION (DIC) METHOD

$$C(u,v)=\frac{\sum_{i=1}^{m}\sum_{j=1}^{m}\left[f\left(x_i,y_j\right)-\bar{f}\right]\left[g\left(x'_i,y'_j\right)-\bar{g}\right]}{\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m}\left[f\left(x_i,y_j\right)-\bar{f}\right]^2}\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m}\left[g\left(x'_i,y'_j\right)-\bar{g}\right]^2}}$$

$$x' = x + u + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

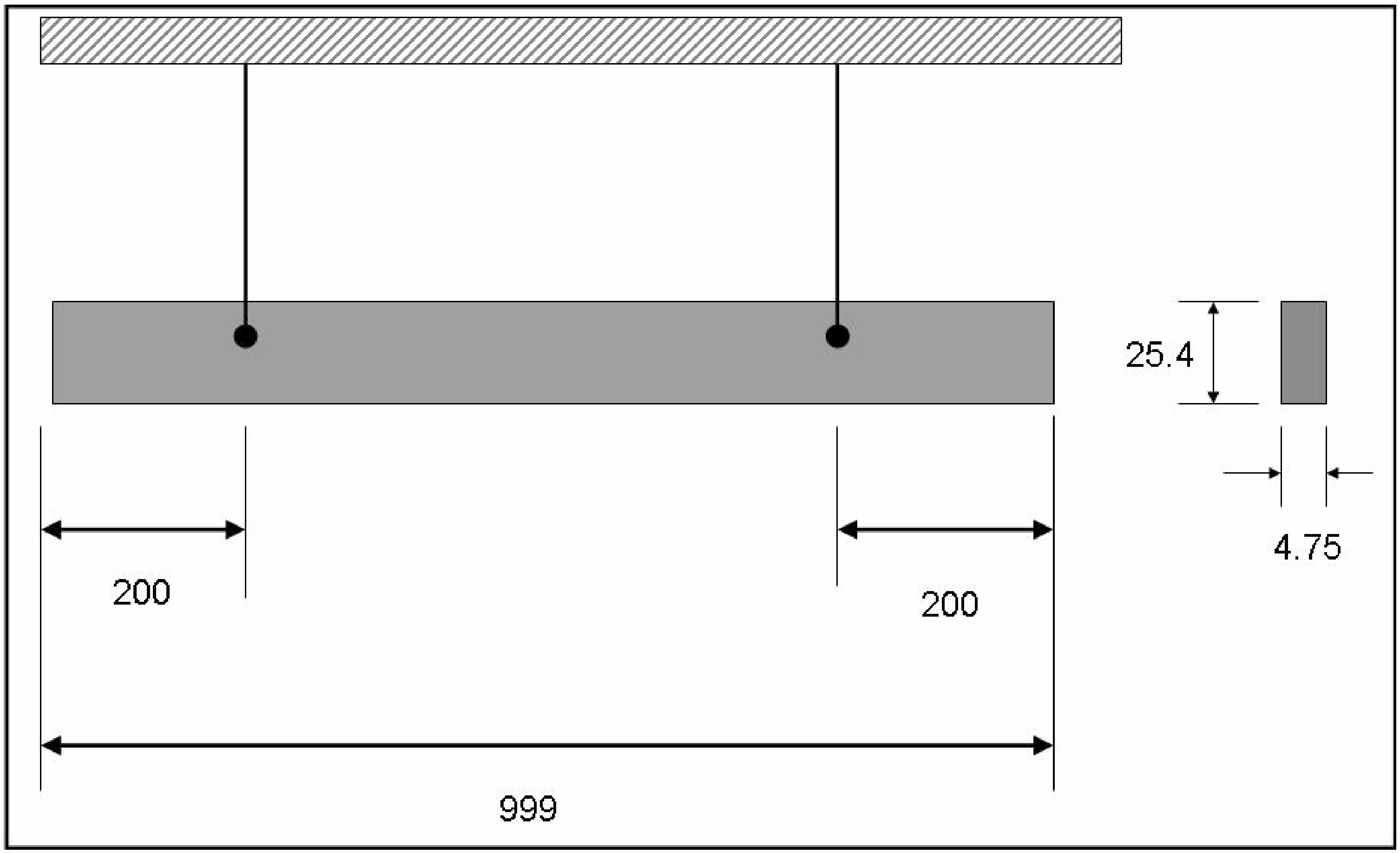
$$y' = y + v + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

EXPERIMENTAL SETUP

- Aluminum beam
- Parameter estimation based on modal analysis
- Parameter estimation based on full field measurement on a subdomainof the beam.

Modal Analysis

- Free-free beam



Modal Analysis

- Natural frequencies

Mode k	f_k (Hz)
1	23.25
2	64.5
3	127.5
4	212
5	317

- Estimated E

Error Function	$\sum_{j=1}^{5}(f_j - f_j^{Exp})$	$\sum_{j=1}^{5}(f_j - f_j^{Exp})/f_j^{Exp}$
$\hat{E}$ **(GPa)**	**68.7**	**68.3**

Table 2. Estimated E for two different error functions.

DIC

DIC

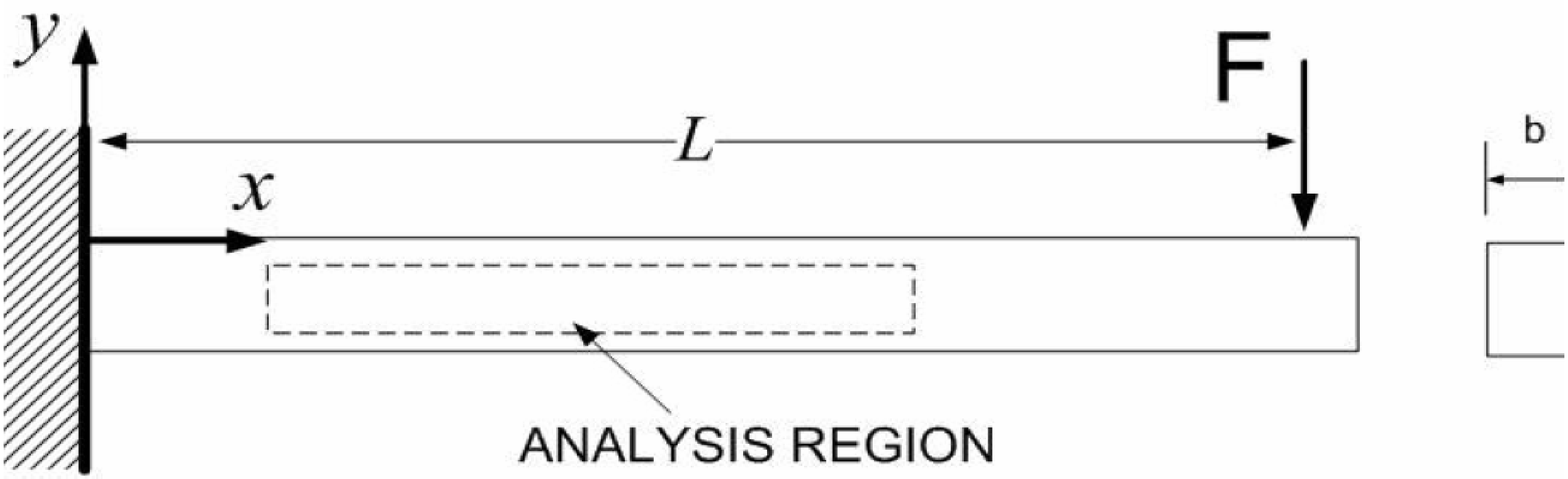
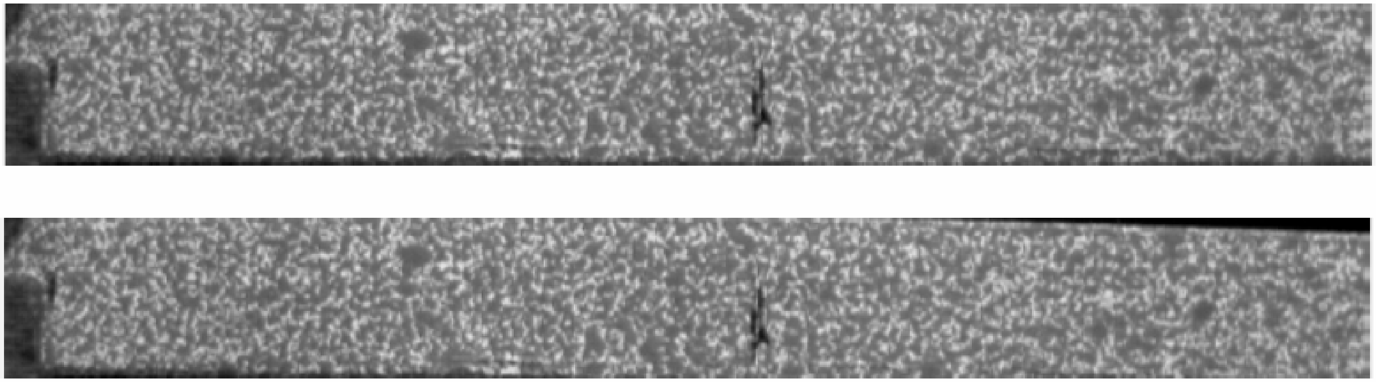


Figure 4. Cantilever beam details.

DIC



(a)

(b)

Figure 5. Pattern of coating specimen, (a) underformed, (b) deformed.

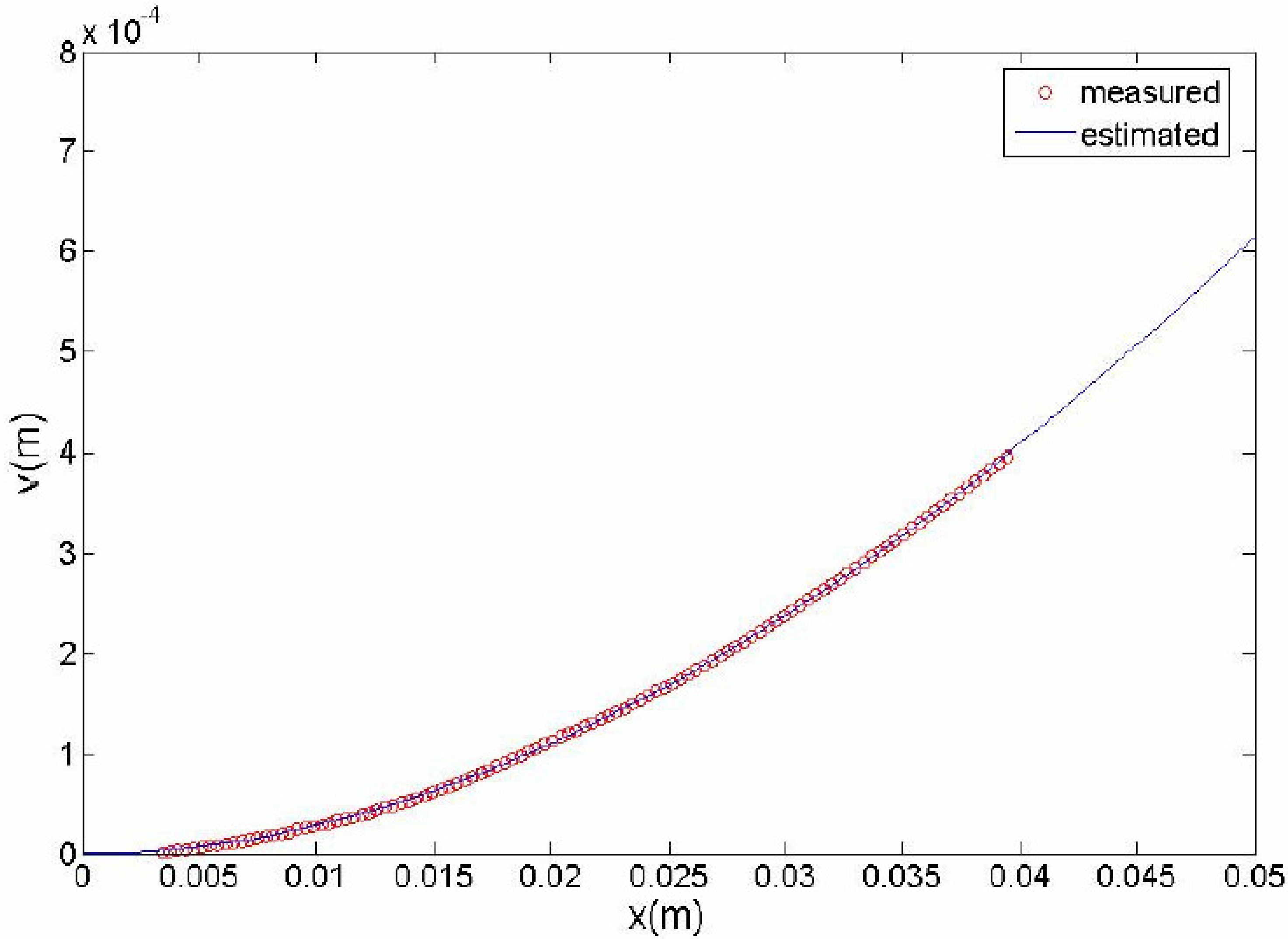
RESULTS

- It was used the Levenberg-Marquardt
- parameter estimation technique considering the following parameterization E = p1x109 and the initial guess

P1 (0) = 30. The provided result is p1 = 49.6 which represents a discrepancy of approximately 27% with respect to the values shown in table (2).

Problem ?

RESULTS



x 10^-4
8
7
6
5
4
3
2
1
0
v(m)
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045
0.05
x(m)
measured
estimated

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