RAPSODEE UMR 2392 CNRS

Curso de Posgrado

Métodos Inversos Aplicados a Ingeniería

Introduction to Inverse Problems:

Analytical data reduction

Olivier Fudym

Mar del Plata, UNMP

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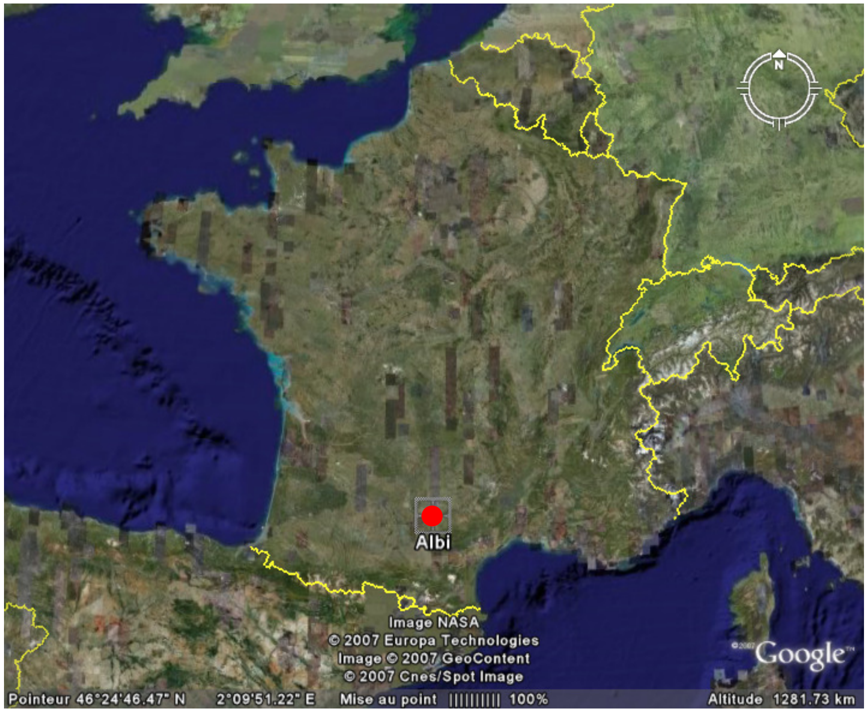
fudym@enstimac.fr



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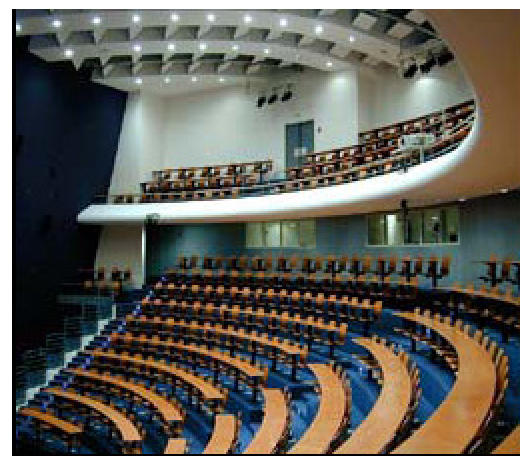
Chemical Engineering Laboratory for Finely Divided Solids, Energy & Environment

- South West of France
- Ministère de l'Economie, des Finances, et de l'Industrie
- Civil Engineers *(Major in Chemical Engineering)*



Inverse Problems: yields from indirect measurements

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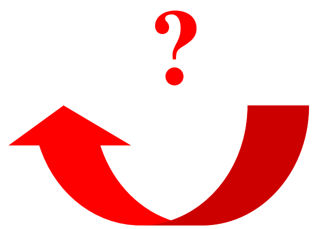
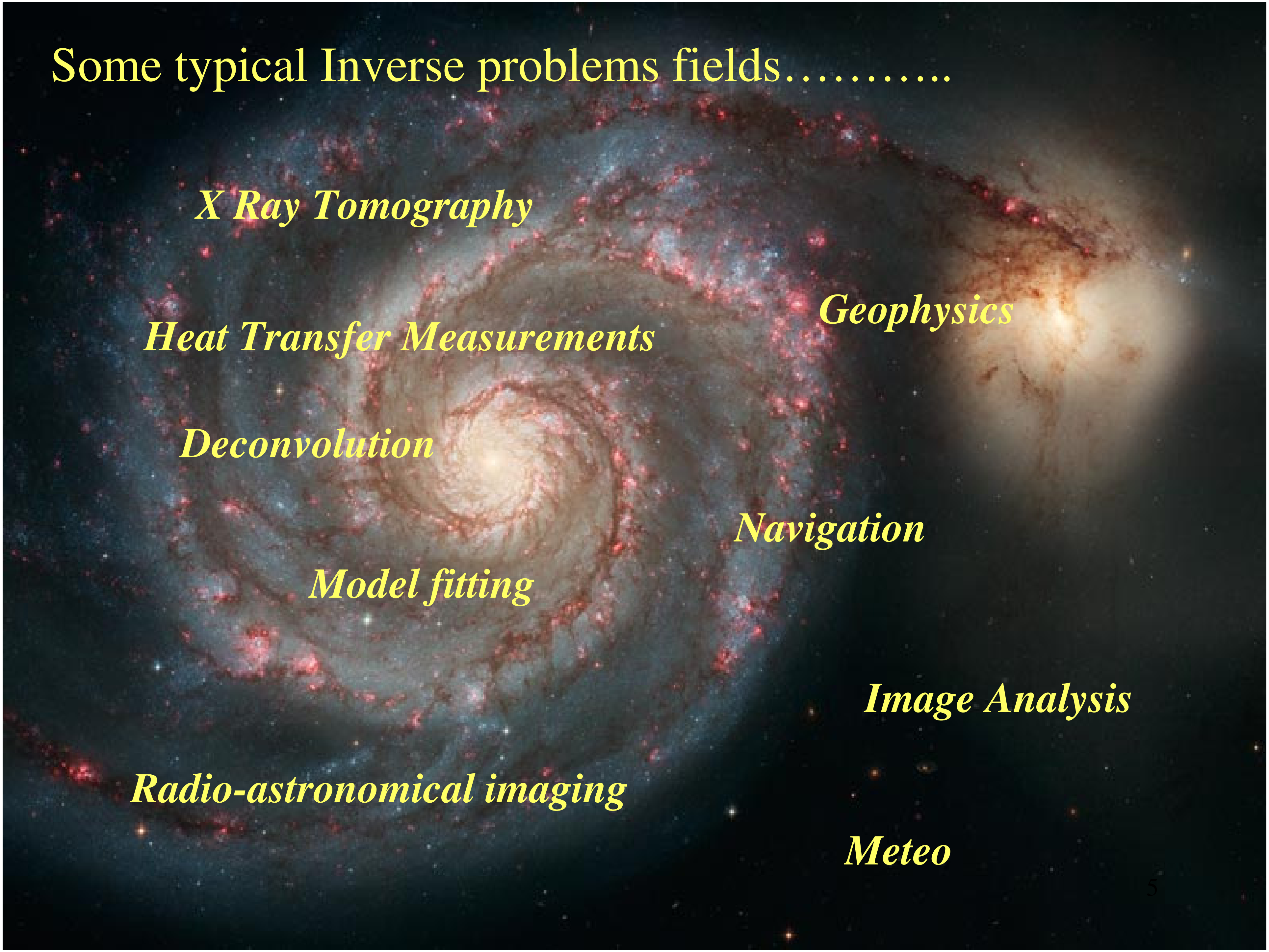
Retrieve some quantities from measured data

Cause



Effect

Some typical Inverse problems fields………..



X Ray Tomography

Heat Transfer Measurements

Geophysics

Deconvolution

Navigation

Model fitting

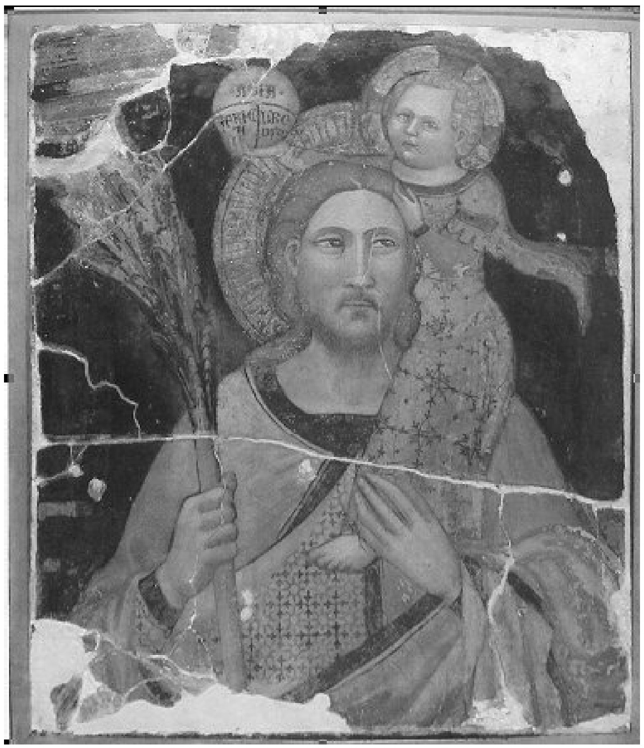
Image Analysis

Radio-astronomical imaging

Meteo

Saint Christopher fresco

Tommaso del Mazza (1385 – 1390) Louvre museum



Infra-red photothermal thermography: A tool of assistance for the restoration of murals paintings?
Jean Charles Candoré, Gabriela Szatanik, J.L Bodnar, Vincent Detalle and Philippe Grossel, QIRT 2006, June 28th-30th, Padova, Italy

Forward mapping

Data space

measurements…

Parameters space

quantities to be retrieved…

$$d = A(f) + n$$

Forward mapping

Measurement noise

$\bar{d} - d$

« True » data : no meas. noise

Type of problems

Forward mapping A = linear ; non-linear

Parameters	Data
continuous	continuous
continuous	discrete
discrete	continuous
discrete	discrete

Well-posed problem, ill-posed problem

Definition of an ill-posed problem, by [Hadamard, 1923]

$$A(f) = d$$

	Well-posed inverse problem
1/ a solution exists	$d \in Im(A)$
2/ the solution is unique	$Ker(A) = \{0\}$
3/ the solution is continuous	A^{-1} continuous

Ill-Posed problem: *if at least one of these conditions is satisfied*

Bad-conditionned problem

Even if the inverse problem is well-posed

small variation of data

$$\frac{\|\Delta f\|}{\|f\|} \leq cond(A)\frac{\|\Delta d\|}{\|d\|}$$

resulting variation of parameters

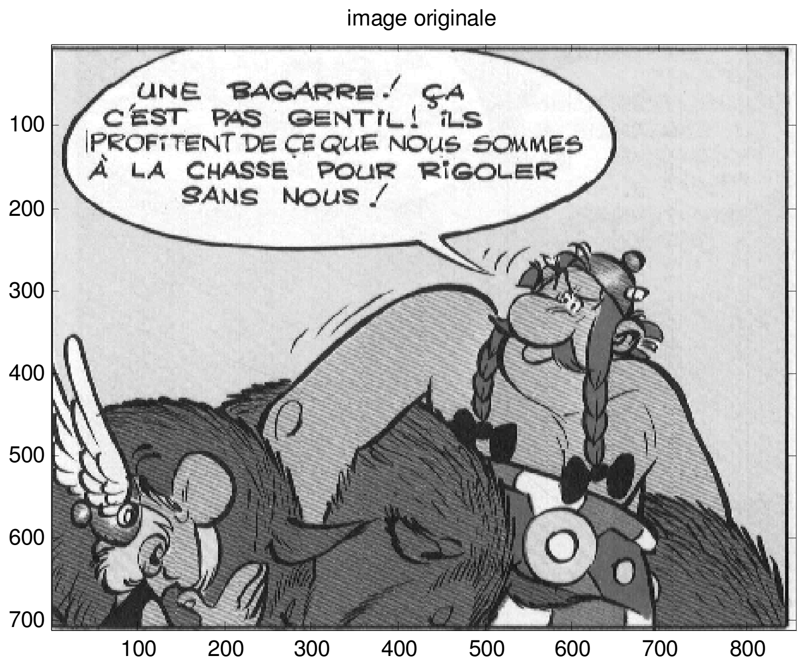
$$cond(A) = \|A\|\left\|A^{-1}\right\|$$

Condition number of A

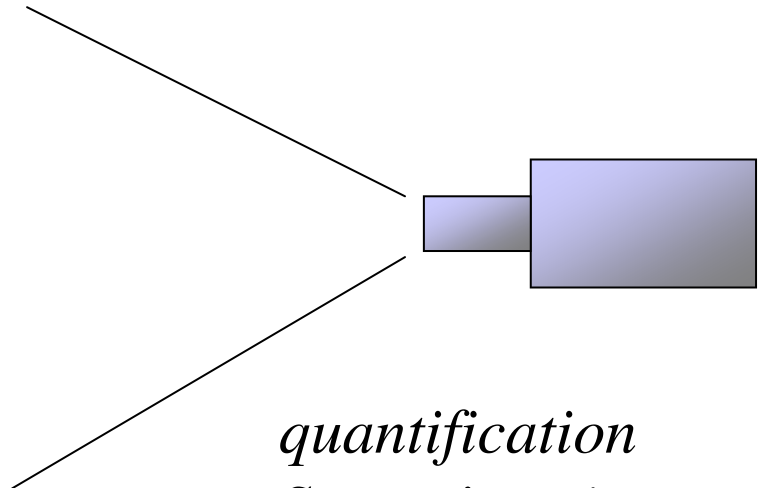
$cond(A) \gg 1$ $\Longrightarrow$ bad-conditionned problem $\Longrightarrow$ $\dfrac{\|\Delta f\|}{\|f\|} \gg \dfrac{\|\Delta d\|}{\|d\|}$

Remark : $cond(A) = 1$ when $A = \alpha I$

Image processing example: deconvolution with FFT



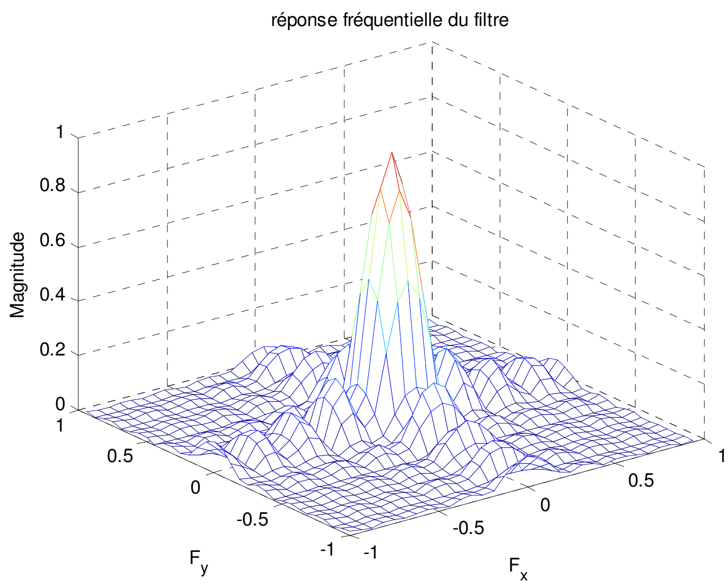
Original Scene



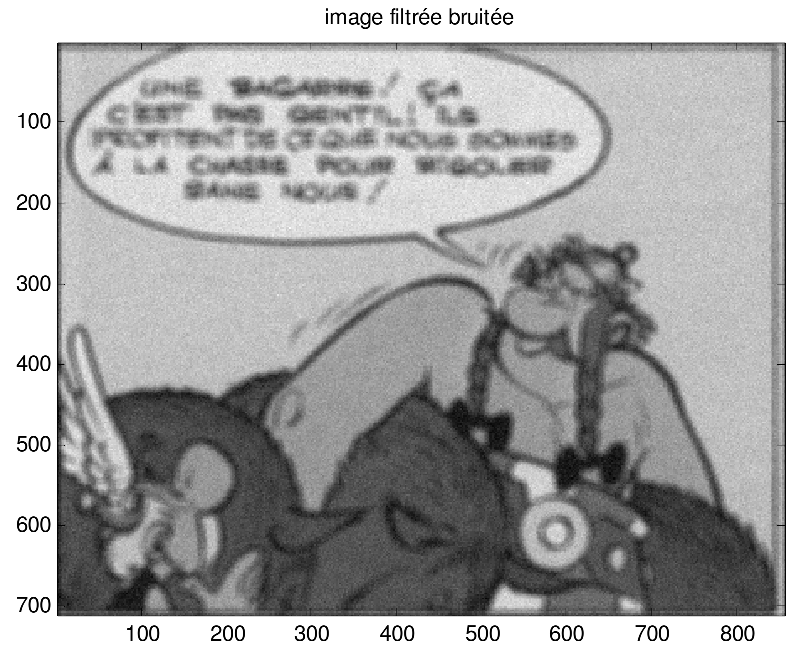
quantification
Sensor's noise
defocusing, …

simulated by
LP filter + Noise

Image processing example: deconvolution with FFT



Spectral response of the filter



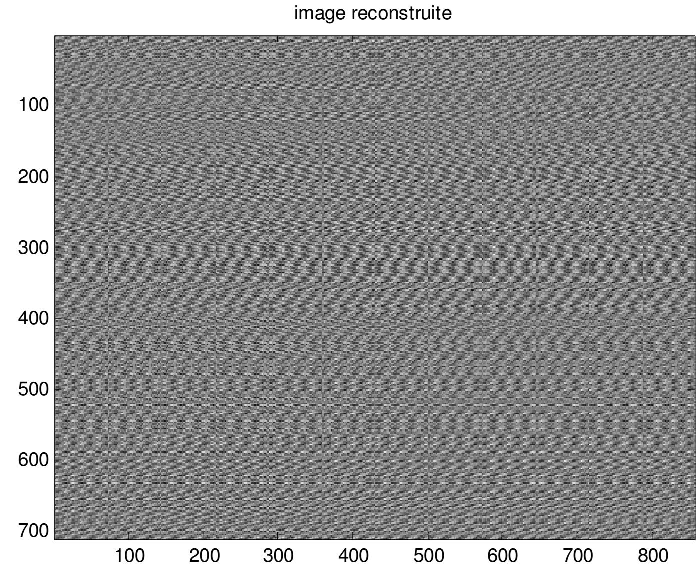
Filtered and noisy image

$$D(r) = X(r)H(r) + N(r)$$

Image processing example: deconvolution with FFT

…you try to retrieve the original scene by:

$$\hat{X}(r) = \frac{D(r)}{H(r)}$$



Retrieved image

WHY ?

In fact, this approach yields:

$$\hat{X}(r) = \frac{X(r)H(r) + N(r)}{H(r)} = X(r) + \frac{N(r)}{H(r)}$$

➔ *Amplification of noise when H(r) tends to zero*

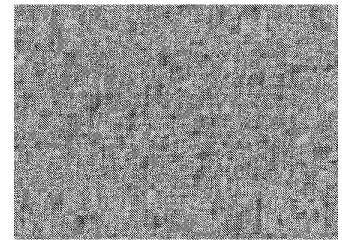
Image processing example: deconvolution with FFT

Regularization

image filtrée bruitée

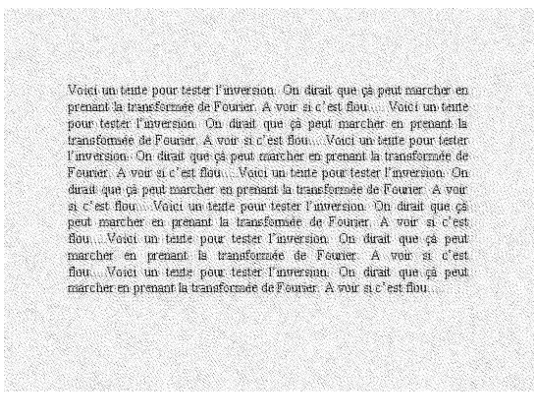
$$\hat{X}_k(r) = X(r) + \frac{N(r)}{H(r) + k}$$

image reconstruite



k = 0.001

image reconstruite



k = 0.01

image reconstruite

k = 1

Derivation of a signal

$$\frac{dT}{dt} = \varphi(t)$$

1/ given $\varphi(t) = 2t$

2/ compute T(t) by intergation: $T(t) = t^2$

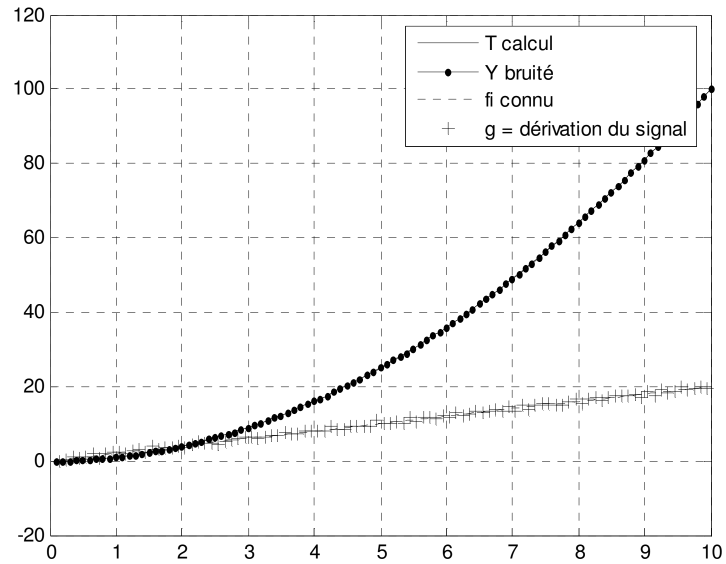
3/ simulate measurements by adding random error:

$$Y(t) = T(t) + e(t)$$

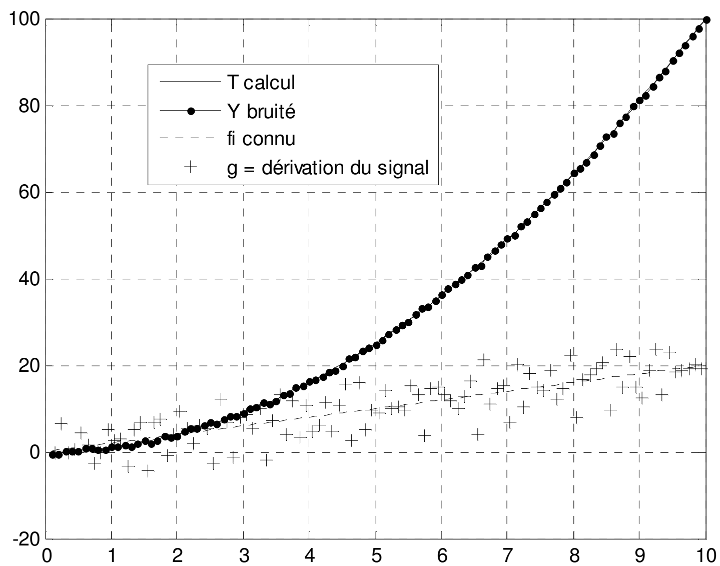
4/ derive the simulated data $g(t) = \frac{dY}{dt}$

Derivation of a signal

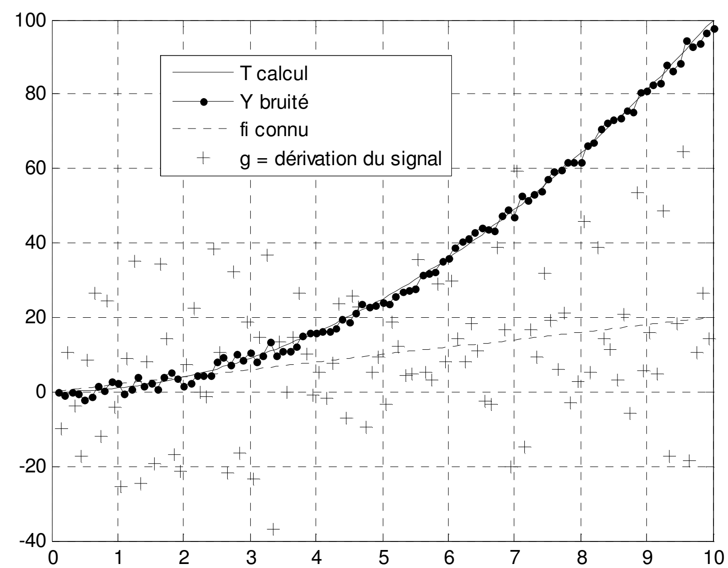
$$\frac{dT}{dt} = \varphi(t)$$



$$\sigma_T = 0.03$$



$$\sigma_T = 0.28$$



$$\sigma_T = 1.4$$

Matlab program: derivation

$$\frac{dT}{dt} = \varphi(t)$$

```
% Derivation d’un signal expérimental
% dT/dt = fi(t)
% T=t**2 ==> fi = 2t
% On choisit fi, pour calculer T
% On simule des données expérimentales Y = T + erreur aléatoire
% g(t) est la dérivation expérimentale de Y
clear;dx=0.1;n=100;x=dx*(1:n);
T=x.*x;
gain=1;
bruit=gain*(0.5-rand(size(x))); % bruit de mesure
Y=T+bruit;                       % Y simule les mesures
fi=2*x;                         % fi calculé
g=diff(Y)/dx;                   % Derivation du signal
x1=dx*(1:n-1);
xr=x1+0.5*dx*ones(size(x1));
figure(gcf);
plot(x,T,'k-',x,Y,'k.-',x,fi,'k:',xr,g,'k+');
legend('T calcul','Y bruité','fi connu',‘g = dérivation signal');
title(‘bruit std = 0.03');
```

Deconvolution of a signal

$$\frac{dT}{dt} = \varphi(t) - hT(t)$$

Solution : $$T(t) = \int_0^t \varphi(t-\tau)\exp(-h\tau)\,d\tau$$

1/ given both input and the impulse response *a(t)*

2/ compute T(t) by convolution

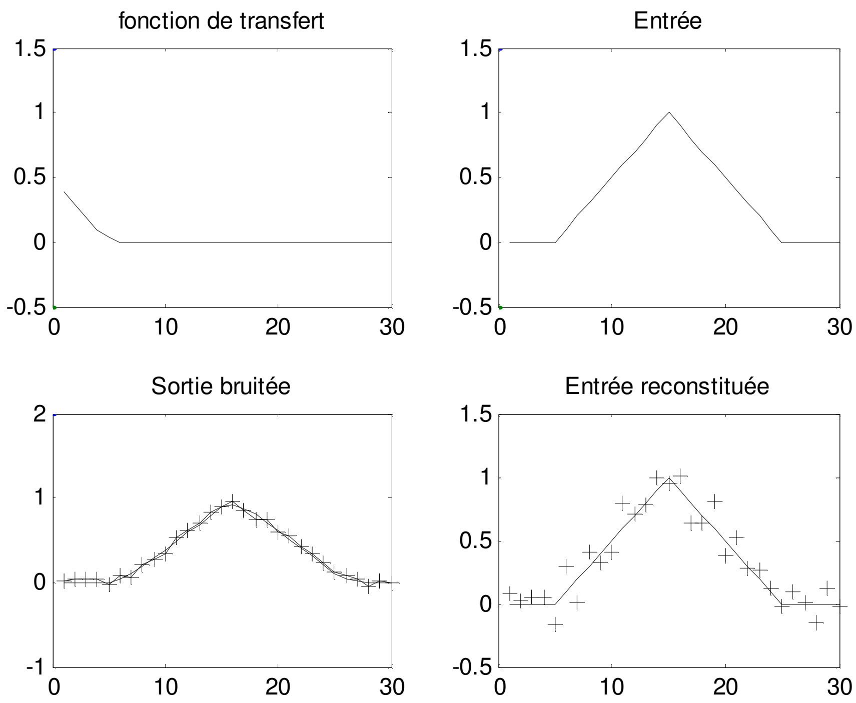
3/ simulate data

$$Y(t) = T(t) + e(t)$$

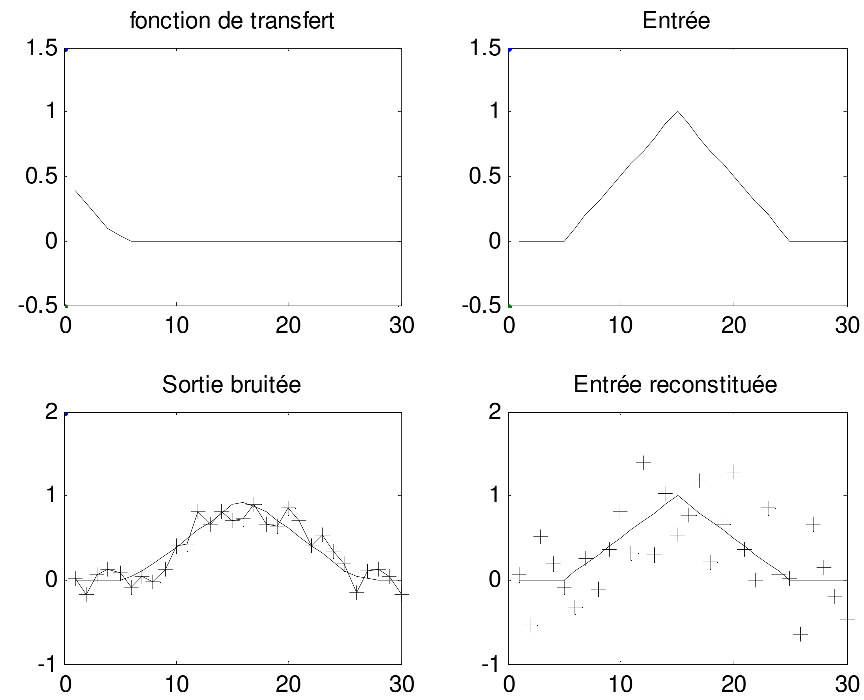
4/ deconvolution de *Y(t)* with *a(t)*

Deconvolution of a signal

$$\frac{dT}{dt} = \varphi(t) - hT(t)$$



$$\sigma_T = 0.03$$



$$\sigma_T = 0.13$$

Matlab program: deconvolution

$$\frac{dT}{dt} = \varphi(t) - hT(t)$$

```
% deconvolution d'un signal experimental
% dT/dt = fi(t) -kT   t=0 T=0
clear;
% Fonction de transfert
a=[1 0.75 0.5 0.25 0.1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 ];
a=(1/sum(a))*a;
n=length(a);
x=(1:n);un=ones(1,(2*n-1));
% Choix de l'entrée
fi=zeros(size(a));fi(6:15)=0.1*(1:10);fi(16:25)=1-0.1*(1:10);

% Convolution
c=conv(a,fi);
gain=0.5;bruit=gain*(0.5*un-rand(size(un)));sigma=std(bruit)
cr=c+bruit;
fir=deconv(cr,a);

subplot(221),plot(x,a,'k',0,1.5,0,-0.5),title('fonction de transfert');
subplot(222),plot(x,fi,'k',0,1.5,0,-0.5),title('Entrée');
subplot(223),plot(x,c(1:n),'k',x,cr(1:n),'k+-',0,2),title('Sortie bruitée');
subplot(224),plot(x,fi,'k',x,fir,'k+'),title('Entrée reconstituée');
```

Example of a bad-conditionned matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1.01 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{\text{inversion}} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1.01 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.01 \end{bmatrix} \xrightarrow{\text{inversion}} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

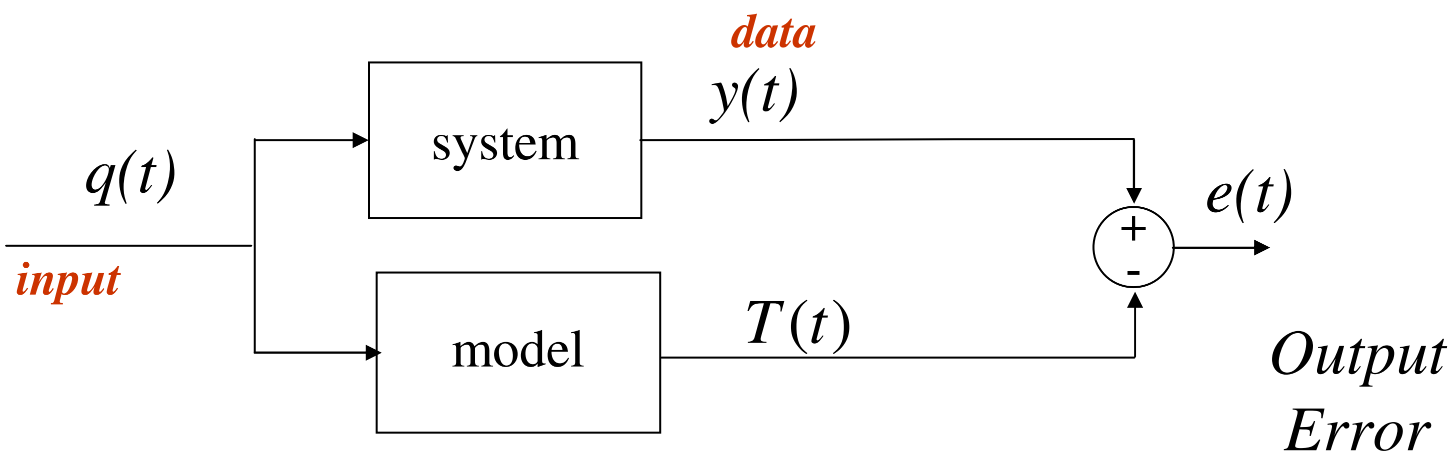
```
% Exemple de Matrice mal conditionnée
clear;
A=[1 1 ; 1 1.01];
Y=[1 1]';
beta=inv(A)*Y

Yr=[1 1.01]';
betar=inv(A)*Yr
% valeurs propres de A
V=eig(A)
% Nombre de conditionnement
cond=max(abs(V))/min(abs(V))
```

eigenvalues

=   0.0050
*   2.0050*

Cond(A) = 402 >> 1

Parameter Estimation



Linear + Discrete Approach $\mathbf{T} = f(x_1, x_2, \ldots x_k, \boldsymbol{\beta})$

$\boldsymbol{\beta} = [\beta_1, \beta_2, \beta_3 \ldots . \beta_p]^t$ Parameters Vector

$(x_1, x_2, \ldots x_k)$ Independant Variables

Linear Parameters Estimation Approach : Ordinary Least Squares

observable → $Y(t_i) = T(t_i) + e_{Y(t_i)}$ ← errors

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$$

$\mathbf{Y}$: ***(n,1)***

$\boldsymbol{\beta}$: ***(p,1)***

$\mathbf{e}$: ***(n,1)*** Random Variable

$$\mathbf{Y} = [Y_1, Y_2, Y_3 .... Y_n]^t$$

$$\boldsymbol{\beta} = [\beta_1, \beta_2, \beta_3 .... \beta_p]^t$$

$$\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 .... \mathbf{X}_p] = \begin{bmatrix} X_{11} & X_{12} & ... & X_{1p} \\ X_{21} & X_{22} & ... & X_{2p} \\ ... & ... & ... & ... \\ X_{n1} & X_{n2} & ... & X_{np} \end{bmatrix}$$

(n,p)

Linear Parameters Estimation Approach : Ordinary Least Squares

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$$

Minimize the norm of output error :

$$S_{OLS} = \mathbf{e}^t\mathbf{e} = (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})^t.(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})$$

Assuming

- zero mean errors
- additive errors
- constant variance
- uncorrelated errors
- X_{ij} known and non stochastics
- parameters non stochastics and no information a priori

(constant variance, uncorrelated errors →) $cov(\mathbf{e}_Y) = \sigma_Y^2 \mathbf{I_p}$

Linear Parameters Estimation Approach : Ordinary Least Squares

$$\nabla_{\beta}.S_{OLS}(\hat{\boldsymbol{\beta}}) = 2\nabla_{\beta}.\left(\mathbf{X}\hat{\boldsymbol{\beta}} - \mathbf{Y}\right)^{t}\left(\mathbf{X}\hat{\boldsymbol{\beta}} - \mathbf{Y}\right) = 0$$

where $\nabla_{\beta}.\left(\mathbf{X}\boldsymbol{\beta} - \mathbf{Y}\right)^{t} = \nabla_{\beta}.\boldsymbol{\beta}^{t}\mathbf{X}^{t} = \mathbf{X}^{t}$ ➔ $\mathbf{X}^{t}\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}^{t}\mathbf{Y}$

OLS Estimator $\hat{\boldsymbol{\beta}}_{OLS} = (\mathbf{X}^{t}\mathbf{X})^{-1}.\mathbf{X}^{t}\mathbf{Y}$

Parameters Covariance Matrix $cov(\mathbf{e}_{\boldsymbol{\beta}}) = (\mathbf{X}^{t}\mathbf{X})^{-1}.\sigma_{Y}^{2}$

Example : Estimation of one parameter

$$\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ \vdots \\ T_N \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_N \end{bmatrix} \beta$$

$$\hat{\beta} = \frac{\sum_{i=1}^{N} f_i \hat{T}_i}{\sum_{i=1}^{N} f_i^{\,2}}$$

$$\sigma_\beta^{\,2} = \frac{\sigma^2}{\sum_{i=1}^{N} f_i^{\,2}}$$

$$\|f(t)\| = \sqrt{\int_0^{t_{max}} f^2(t)\,dt}$$

$$\sigma_\beta^{\,2} = \frac{t_{max}\,\sigma^2}{N\|f\|^2}$$

- If N is small, T_i must be chosen for f maximum

- If T_i equally distributed ➔ N must be as high as <u>possible !</u>

Example : Estimation of two parameters

$$\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix} = \underbrace{\begin{bmatrix} f_1 & g_1 \\ f_2 & g_2 \\ \vdots & \vdots \\ f_N & g_N \end{bmatrix}}_{\boldsymbol{X}} \cdot \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

$$\hat{\beta}_1 = \frac{\left(\boldsymbol{f}^t \cdot \hat{\boldsymbol{T}}\right)}{\left(\boldsymbol{f}^t \cdot \boldsymbol{f}\right)} \qquad \hat{\beta}_2 = \frac{\left(\boldsymbol{g}^t \cdot \hat{\boldsymbol{T}}\right)}{\left(\boldsymbol{g}^t \cdot \boldsymbol{g}\right)}$$

$$\operatorname{cov}\left(\hat{\boldsymbol{B}}\right) = \sigma^2 \begin{pmatrix} \left(\boldsymbol{f}^t \boldsymbol{f}\right)^{-1} & 0 \\ 0 & \left(\boldsymbol{g}^t \boldsymbol{g}\right)^{-1} \end{pmatrix}$$

$$\operatorname{cov}\left(\hat{\boldsymbol{B}}\right) \approx \sigma^2 \left(\frac{N}{t_{\max}}\right)^{-1} \begin{pmatrix} \|f\|^{-2} & 0 \\ 0 & \|g\|^{-2} \end{pmatrix} \qquad \operatorname{cond}\left(\operatorname{cov}\left(\hat{\boldsymbol{B}}\right)\right) \approx \frac{\|f\|^2}{\|g\|^2}$$

- If T_i equally distributed ➔ N must be as high as <u>possible !</u>

- The condition number is <u>independent of N</u> !

Sensitivity Coefficients

$$\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 .... \mathbf{X}_p] = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1p} \\ X_{21} & X_{22} & \cdots & X_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ X_{n1} & X_{n2} & \cdots & X_{np} \end{bmatrix}$$

Sensitivity matrix

$$X_{ij} = X_j(t_i, \boldsymbol{\beta}) = \left.\frac{\partial Y}{\partial \beta_j}\right|_{t_i, \boldsymbol{\beta}}$$

Linear Estimation: if X_{ij} do not depend on the parameters

Gauss Markov Estimator

Assuming

- zero mean errors
- additive errors
- covariance matrix known
- X_{ij} known and non stochastics
- parameters non stochastics and no information a priori

$$\mathrm{cov}(\mathbf{e}_{\mathbf{Y}}) = \sigma^2\Omega$$

$$S_{GM} = \mathbf{e}^t\mathbf{e} = (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})^t . \left(cov(\mathbf{e}_{\mathbf{Y}})\right)^{-1} . (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})$$

$$\hat{\boldsymbol{\beta}}_{GM} = (\mathbf{X}^t\Omega^{-1}\mathbf{X})^{-1} . \mathbf{X}^t\Omega^{-1}\mathbf{Y}$$

$$cov(\mathbf{e}_{\boldsymbol{\beta}}) = (\mathbf{X}^t\Omega^{-1}\mathbf{X})^{-1} . \sigma^2$$

Example: GM estimator for 2 parameters

```
% Estimateur de Gauss Markov
%  y = ax + b
% yr = y + bruit(x)
clear;
n=50;un=ones(size(1:n));x=(1:n);
a=1;
b=0.5;
% bruit d'amplitude variable
gain=0.001;
amp=gain*(x.^3);
bruit=amp.*(0.5-rand(size(x)));
% droites
y=a*x+b;
yr=y+bruit;
% matrice de sensibilité
X=[x' un'];
```

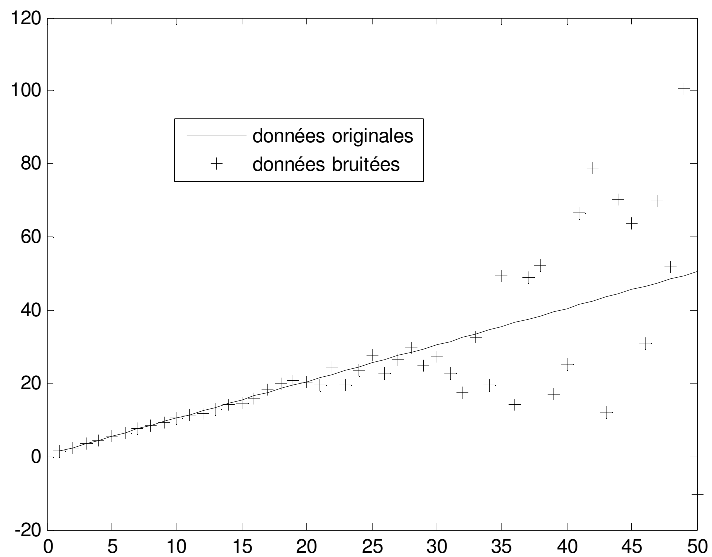
$$Y = ax + b$$

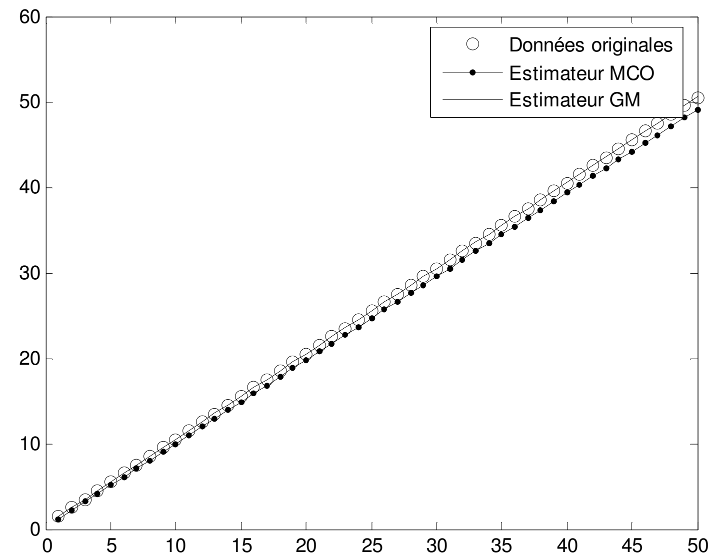
Uncorrelated errors Ω = diagonal matrix

non uniform standard deviation depending on amplitud

```
% inverse de la matrice de covariance
phi=diag(un./amp.^2);
% Estimateur de Gauss Markov (GM)
bgm=inv(X'*phi*X)*X'*phi*yr'
ygm=bgm(1)*x+bgm(2);
% Estimateur MCO
bmc=inv(X'*X)*X'*yr'
ymc=bmc(1)*x+bmc(2);
figure(1), plot(x,y,x,yr,'+')
figure(2),plot(x,y,'ko',x,ymc,'k:',x,ygm,'k-')
```

Example: GM estimator for 2 parameters





$$\boldsymbol{\beta}_0 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \qquad \boldsymbol{\beta}_{OLS} = \begin{bmatrix} 1.060 \\ -1.057 \end{bmatrix} \qquad \boldsymbol{\beta}_{GM} = \begin{bmatrix} 0.9991 \\ 0.5012 \end{bmatrix}$$

GM : gives more strength to the least dispersed measurements

Maximum Likelihood Estimator

Assuming

- errors are zero mean
- errors are additive
- **normally distributed**
- known covariance matrix
- X_{ij} known and non stochastics
- no prior information about the parameters

$$\mathrm{cov}(\mathbf{e_Y}) = \psi = \sigma^2\Omega$$

$$\pi(\mathbf{Y}|\boldsymbol{\beta}) = (2\pi)^{-n/2}|\psi|^{-1} exp(-(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta})^t.\psi^{-1}.(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta})/2)$$

$$S_{ML} = \mathbf{e}^t\mathbf{e} = (\mathbf{Y}-\mathbf{X}\boldsymbol{\beta})^t.\psi^{-1}.(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta})$$

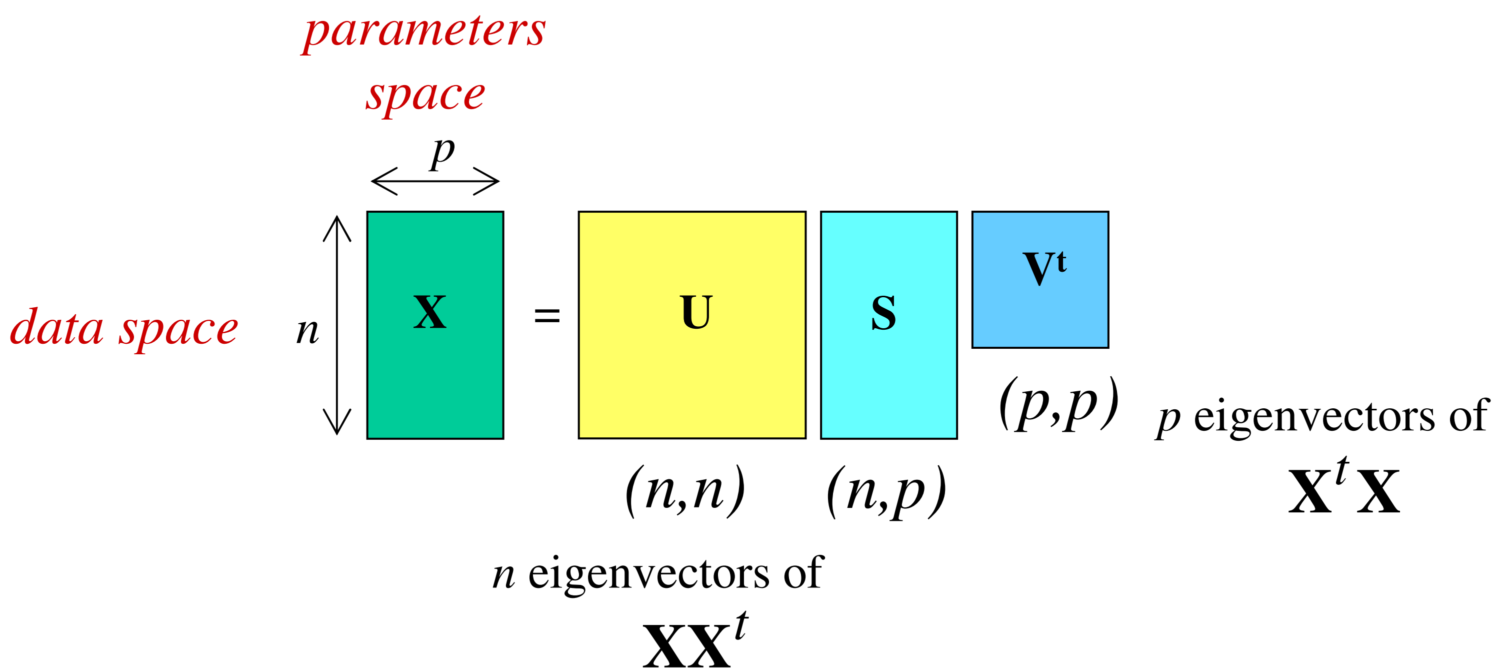
$$\hat{\boldsymbol{\beta}}_{ML} = (\mathbf{X}^t\Omega^{-1}\mathbf{X})^{-1}.\mathbf{X}^t\Omega^{-1}\mathbf{Y}$$

$$cov(\mathbf{e}_{\boldsymbol{\beta}}) = (\mathbf{X}^t\Omega^{-1}\mathbf{X})^{-1}.\sigma^2$$

Spectral analysis of the finite linear problem

$$\mathbf{X}\beta = \mathbf{Y}$$

Singular Value *Decomposition* (SVD) $\mathbf{X} = \mathbf{USV}^t$



Spectral analysis of the finite linear problem

$$\mathbf{USV^t\beta = Y} \quad \mathbf{U^tUSV^t\beta = U^tY} \quad \mathbf{SV^t\beta = U^tY}$$

$$\mathbf{Sb = Z}$$

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & \lambda_2 & & & & 0 \\ & 0 & \dots & & & 0 \\ \dots & & & \lambda_r & & \\ & & & \dots & 0 & \dots \\ & & & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_r \\ \dots \\ b_p \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_r \\ \dots \\ \dots \\ z_n \end{bmatrix}$$

No solution if : r < n

No unicity if si : r < p

OLS Estimator : $\hat{\mathbf{b}} = \left(\mathbf{S^tS}\right)^{-1}\mathbf{S^tZ}$

$$\hat{\mathbf{b}} = \sum_{i=1}^{r} \frac{z_i}{\lambda_i}\mathbf{V_i} + \sum_{i=r+1}^{p} \tilde{b}_i \mathbf{V_i}$$

Stability and condition number

perturbation on the k^{th} component $\delta\mathbf{Z} = \delta z_k \mathbf{U_k}$

➔ perturbation on **b** $\delta\hat{\mathbf{b}} = \dfrac{\delta z_k}{\lambda_k}\mathbf{V_k}$

Relative variation $$\frac{\left\|\delta\hat{\mathbf{b}}\right\|}{\left\|\delta\mathbf{Z}\right\|} = \frac{1}{\lambda_k}$$

uniform perturbation ➔ $$\frac{\delta b_r}{\delta b_1} = \frac{\lambda_1}{\lambda_r}$$ $$cond(\mathbf{X}) = \frac{\lambda_1}{\lambda_r}$$

Regularization = improve the condition number of X

- *Truncature* *Threshold effect*

$$cond(\mathbf{X}_f) = \frac{\lambda_1}{\lambda_f} \qquad \hat{\mathbf{b}}_f = \sum_{i=1}^{f} \frac{z_i}{\lambda_i} \mathbf{V_i}$$

- *Penalization* *Tikhonov*

$$S_\mu(\beta) = \left\| \mathbf{X}\beta - \mathbf{Y} \right\|^2 + \mu \left\| \beta - \tilde{\beta} \right\|^2$$

$$\hat{\beta} = \left(\mathbf{X}^\mathbf{t}\mathbf{X} + \mu \mathbf{I} \right)^{-1} \mathbf{X}^\mathbf{t}\hat{\mathbf{Y}}$$

Regularization = improve the condition number of X

- *Penalization by Tikhonov regularization*

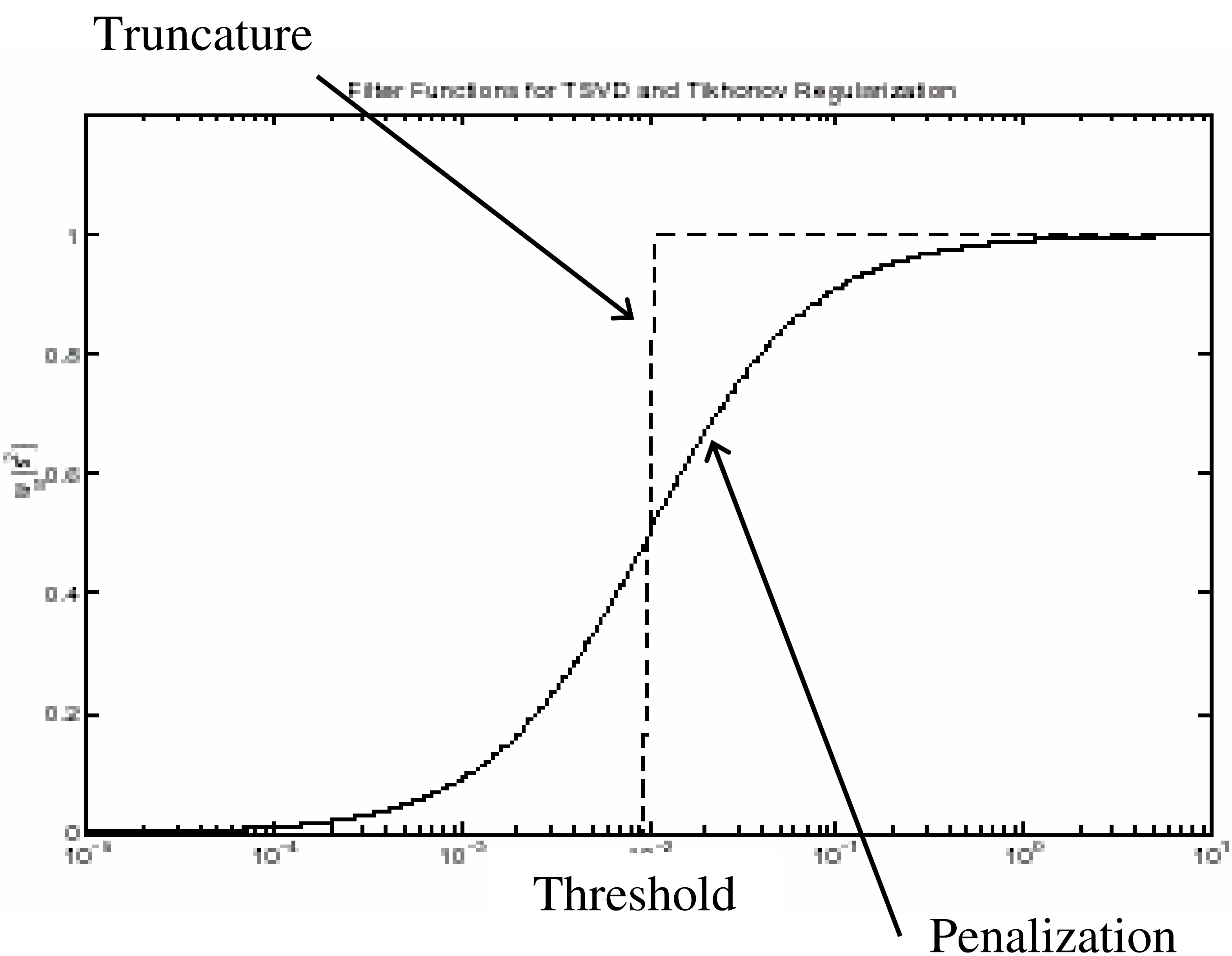
$$S_\mu(\mathbf{b}) = \left\|\mathbf{Sb}-\mathbf{Z}\right\|^2 + \mu\left\|\mathbf{b}-\tilde{\mathbf{b}}\right\|^2$$

$$\hat{\mathbf{b}}_\mu = \left[\mathbf{S}^t\mathbf{S}+\mu\mathbf{I}\right]^{-1}\left(\mathbf{S}^t\mathbf{Z}+\mu\tilde{\mathbf{b}}\right)$$

Full rank r = p

$$\tilde{\mathbf{b}} = 0$$

$$\hat{\mathbf{b}} = \begin{bmatrix} \dfrac{\lambda_1 z_1}{\lambda_1^{\,2}+\mu} \\ \cdot \\ \cdot \\ \dfrac{\lambda_p z_p}{\lambda_p^{\,2}+\mu} \end{bmatrix}$$


Truncature
Filter Functions for TSVD and Tikhonov Regularization
1
0.8
0.6
0.4
0.2
0
Threshold
Penalization

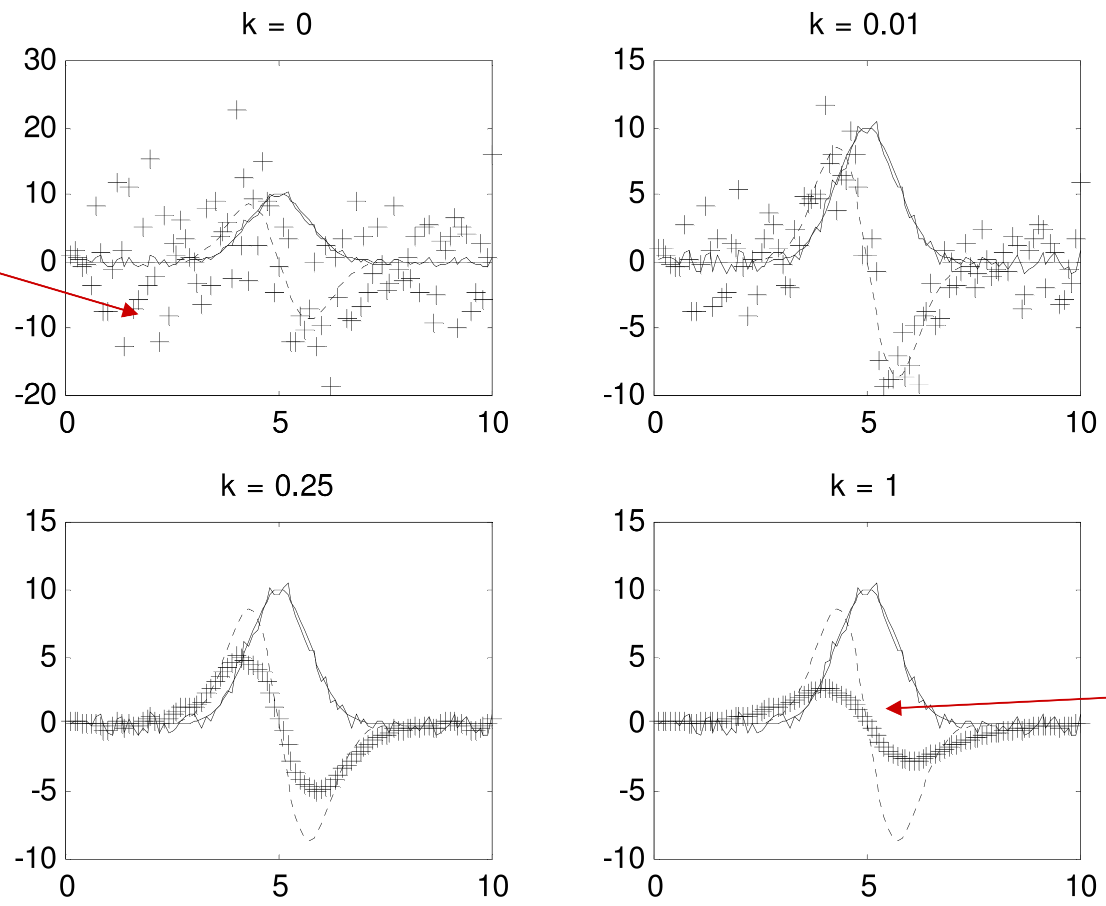
Derivation and inversion with Tikhonov regularization

```
% T=T0 exp(-(t-t0)**2
% fi=dT/dt
% Tr = T + bruit
N=100;dt=0.1;t0=dt*N/2;t=dt*(1:N);T0=10;
T=T0*exp(-(t-t0).^2);fi=-2*(t-t0).*T;
Q=sqrt(T*T');
Amp=20;
bruit=Q/Amp*(0.5-rand(size(t))); % bruit
sigma2=cov(bruit)
Tr=T+bruit;
% Coefficients de régularisation
k=[0 0.01 0.25 1];
% Matrice de sensibilité
X=dt*toeplitz(ones(1,N),[1 zeros(1,N-1)]);
for i=1:length(k)
   G=inv(X'*X+k(i)*eye(N));
   fir=G*X'*Tr';
   VI=eig(G);
   ki=k(i)
   cond_i=max(abs(VI))/min(abs(VI))
   p=i;
   subplot(2,2,p),plot(t,T,'k',t,Tr,'k',t,fi,'k:',t,fir,'k+'),
   title(['k = ',num2str(k(i))])
end
```

Derivation and inversion with Tikhonov regularization

No stabilization effect

$$\sigma^2 = 0.23$$



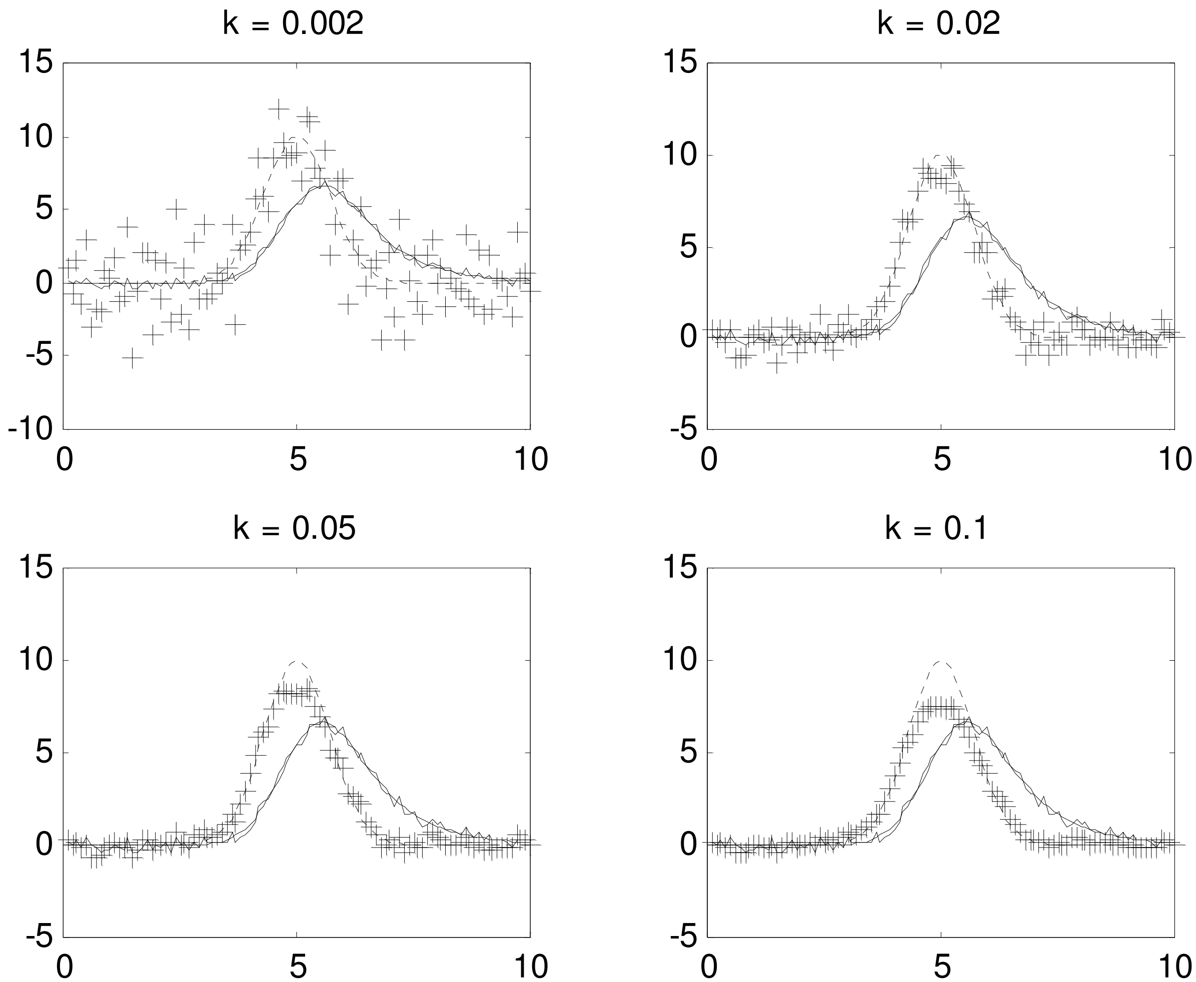
Important bias

Deconvolution and inversion with Tikhonov regularization

```
% dT/dt = fi - hT
% T=conv(fi,exp(-ht))
% Tr = T + bruit
N=100;dt=0.1;t0=dt*N/2;t=dt*(1:N);
fi0=10;h=1;
fi=fi0*exp(-(t-t0).^2);

Q=sqrt(fi*fi');Amp=40;
bruit=Q/Amp*(0.5-rand(size(t))); % ruido
cov(bruit)
k=[0.002 0.02 0.05 0.1]; % Coef de regularización
X=dt*toeplitz(exp(-h*(t(1:N))), zeros(1,N)); % Matriz de sensibilidad
T=X*fi'; % Modelo directo obtenido por convolución
Tr=T'+bruit;
for i=1:length(k)
   G=inv(X'*X+k(i)*eye(N));
   fir=G*X'*Tr';
   VI=eig(G);
   ki=k(i)
   cond_i=max(abs(VI))/min(abs(VI))
   p=i;
   subplot(2,2,p),plot(t,T,'k',t,Tr,'k',t,fi,'k:',t,fir,'k+',0,15,0,-5),
   title(['k = ',num2str(k(i))])
   figure(gcf);
end
```

Deconvolution and inversion with Tikhonov regularization

$\sigma^2 = 0.06$



Regularization Coefficient

$$\mathrm{cov}(e_{\mathbf{Z}}) = \mathbf{U}^{\mathbf{t}}\,\mathrm{cov}(e_{\mathbf{Y}})\mathbf{U}$$

if $\mathrm{cov}(e_{\mathbf{Y}}) = \sigma^2\mathbf{I}$ ⇒ $\mathrm{cov}(e_{\mathbf{Z}}) = \mathbf{U}^{\mathbf{t}}\sigma^2\mathbf{I}\mathbf{U} = \sigma^2\mathbf{I}$

With no regularization :

$$\mathrm{cov}(\hat{\mathbf{b}}) = \sigma^2\left(\mathbf{S}^{\mathbf{t}}\mathbf{S}\right)^{-1} = \begin{bmatrix} \dfrac{\sigma^2}{\lambda_1^2} & . & 0 \\ . & . & . \\ 0 & . & \dfrac{\sigma^2}{\lambda_p^2} \end{bmatrix}$$

Amplification effect due to the « small » eigenvalues On the STD

but: What is « small » ?

Discrepancy principle regularization

If you choose : $\mu = \sigma^2$ ➡

$$\hat{\mathbf{b}} = \begin{bmatrix} \dfrac{\lambda_1 z_1}{\lambda_1{}^2 + \sigma^2} \\ . \\ . \\ \dfrac{\lambda_p z_p}{\lambda_p{}^2 + \sigma^2} \end{bmatrix}$$

and

$$\mathrm{cov}(\hat{\mathbf{b}}) = \sigma^2 \left(\mathbf{S}^{\mathbf{t}}\mathbf{S}\right)^{-1} = \begin{bmatrix} \dfrac{\sigma^2}{\lambda_1^2 + \sigma^2} & . & 0 \\ . & . & . \\ 0 & . & \dfrac{\sigma^2}{\lambda_p^2 + \sigma^2} \end{bmatrix}$$

Discrepancy principle regularization

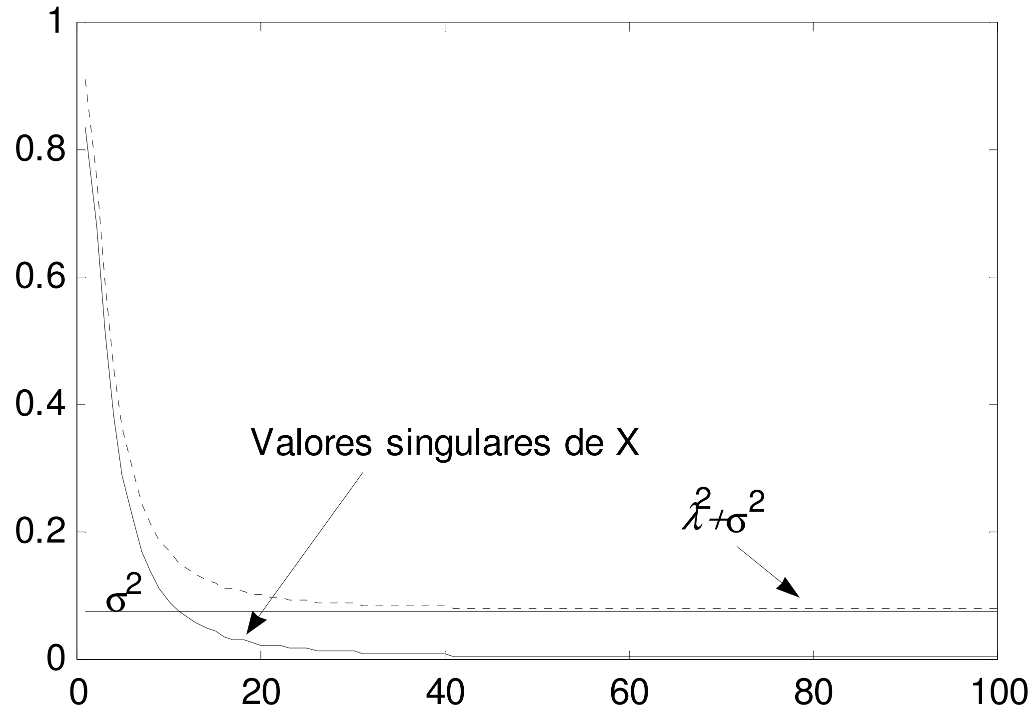
$$\hat{\mathbf{b}} = \begin{bmatrix} \dfrac{\lambda_1 z_1}{\lambda_1^{\ 2} + \sigma^2} \\ . \\ . \\ \dfrac{\lambda_p z_p}{\lambda_p^{\ 2} + \sigma^2} \end{bmatrix}$$

bias for the eigenvalues smallest than the measurement noise std σ

$$\lambda_i >> \sigma \Rightarrow \hat{b}_i \approx \frac{z_i}{\lambda_i} \quad et \quad \mathrm{var}(b_i) \approx \frac{\sigma^2}{\lambda_i^2}$$

$$\lambda_i << \sigma \Rightarrow \hat{b}_i \approx \frac{\lambda_i z_i}{\sigma^2} \quad et \quad \mathrm{var}(b_i) \approx 1$$

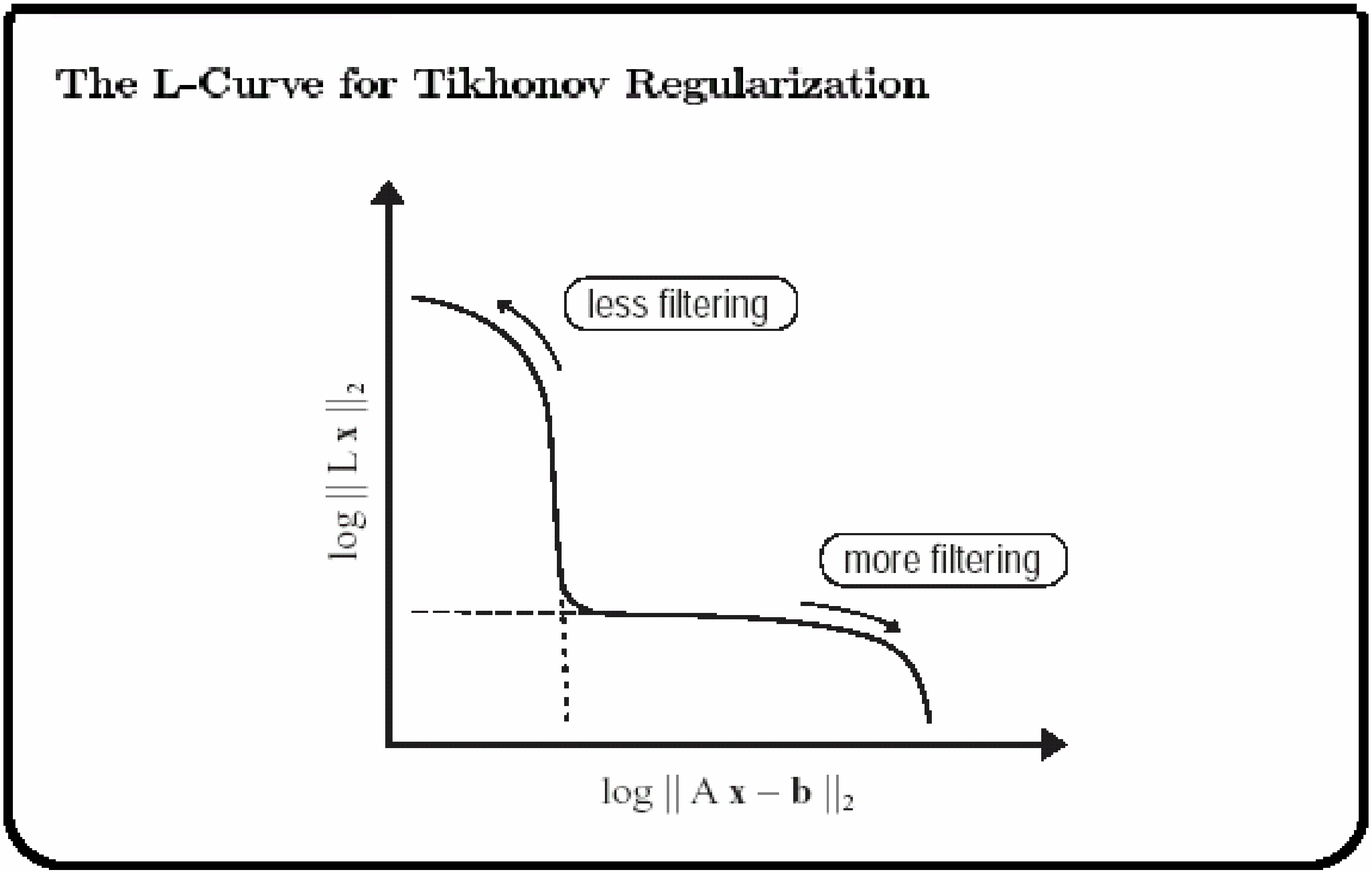
Discrepancy principle regularization



Discrepancy principle regularization

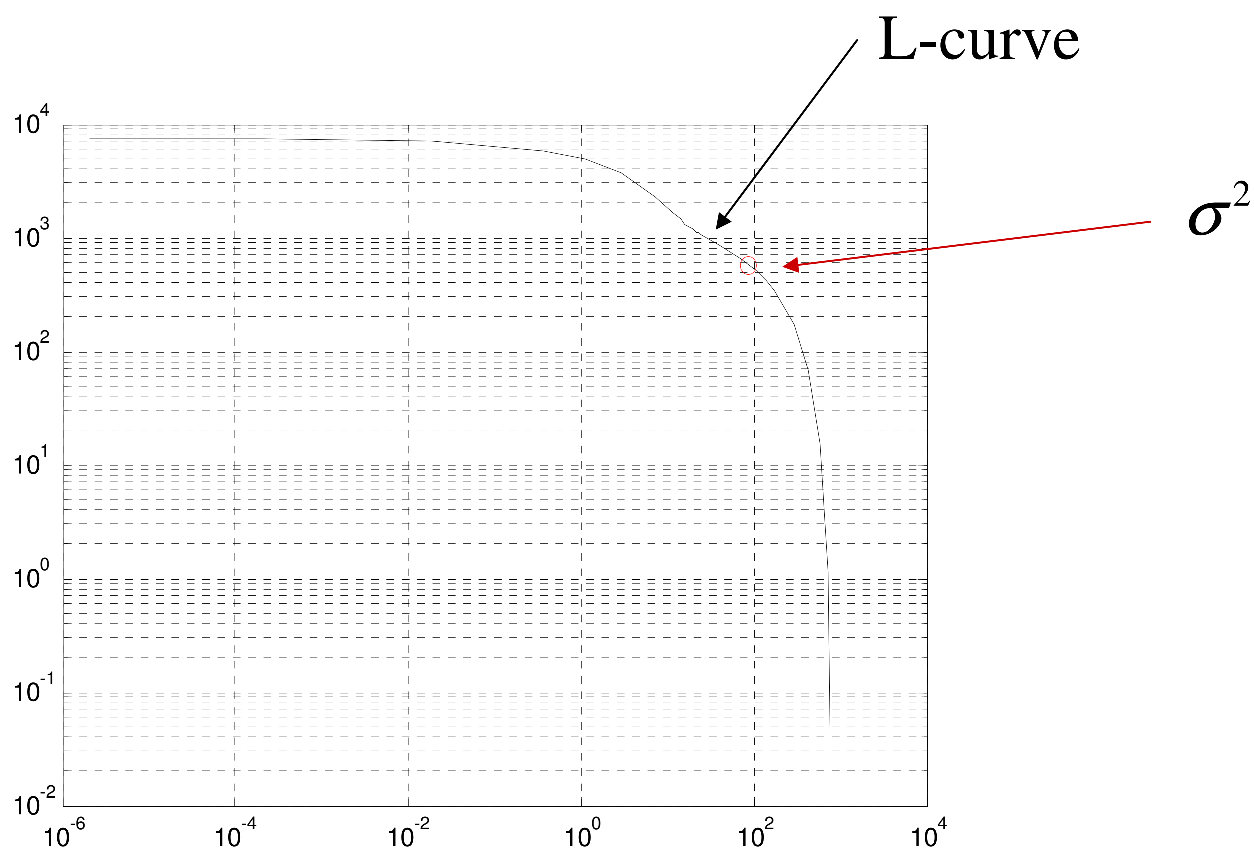
```
N=100;dt=0.1;t0=dt*N/2;t=dt*(1:N);
fi0=10;h=1;
fi=fi0*exp(-(t-t0).^2);
UN=(1:N);gain=1;
ruido=gain*(0.5-rand(size(t))); % ruido
sigy=cov(ruido);
k=sigy;
X=dt*toeplitz(exp(-h*(t(1:N))), zeros(1,N)); % Matriz de sensibilidad
T=X*fi'; % Modelo directo obtenido por convolución
Tr=T'+ruido;
[W,D,V]=svd(X);
DD=diag(D'*D);
DDP=DD+k;
XTX=V*diag(DDP)*V';
figure(1)
plot(UN,DD,'k',UN,DDP,'k:',[1 N],[sigy sigy],'k');
fir=inv(XTX)*X'*Tr';
figure(2),plot(t,T,'k:',t,Tr,'k',t,fi,'k',t,fir,'k+')
```

Choice of the Tikhonov regularization coefficient: The L-curve



Choice of the Tikhonov regularization coefficient: The L-curve

$$\log(\|\hat{\varphi}\|^2)$$



$$\log(\|T_r - X\hat{\varphi}\|^2)$$

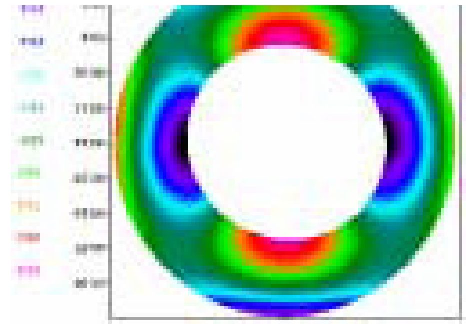
II/ Velocity and heat transfer parameters mapping: infrared image processing

1. Introduction: IR thermography and Parameters mapping
2. Field Estimation for Local Mapping
3. Macroscopic characterization from averaging
4. Modal approach (SVD)

Background : New needs for full fields methods…

Spectacular Recent Progress

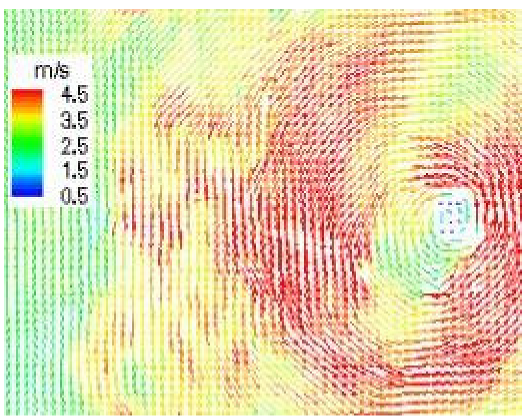
in field measurements…



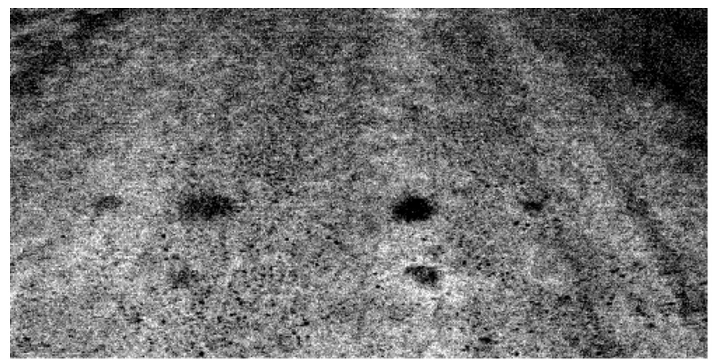
Vibrations Brake disks (ESPI)

Dr. Ettemeyer GmbH & Co, Germany

…Velocity, concentration, déformations, stress, temperature, density…



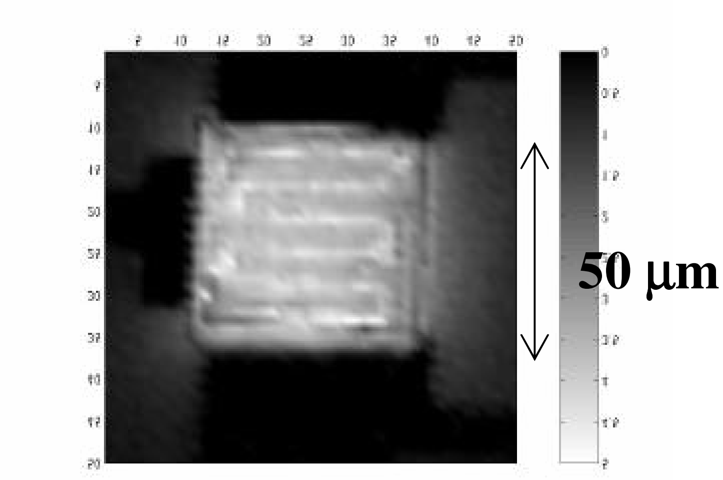
Particule Image Velocimetry(PIV)



Underground Landmines Detection from Thermal Imaging

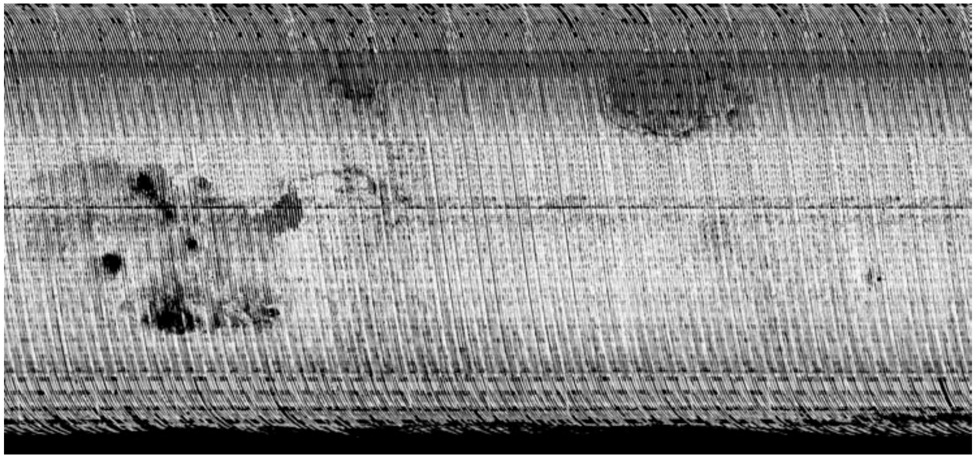
Background…

…From microscopic scale…



Temperature mapping of a micro heater
From photoreflectance imaging

…to astrophysic scale !



Mars Global Surveyor Thermal Emission Spectrometer
Thermal inertia mapping of Mars ground

...Background…

That is the « Full field methods » revolution...

Images are recorded from either :

1. One sensor scanning

Atomic force Microscopy, SThM, photoreflectance imaging, old fashion IR…

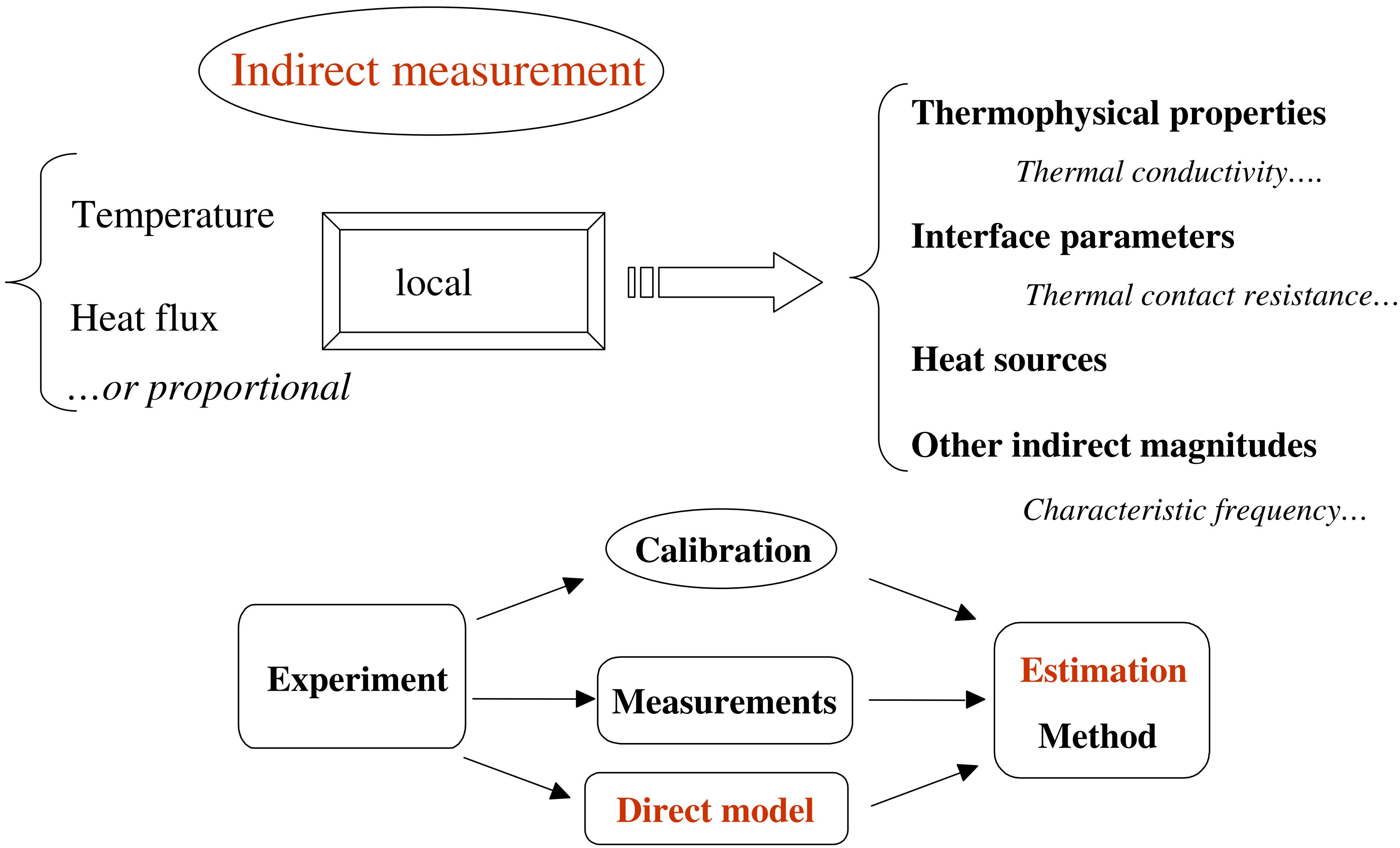
2. From a very high number of spatially distributed sensors

Focal Plane Array camera, PIV, Tomography X…

Challenge : how to process this intensive flux of data ?

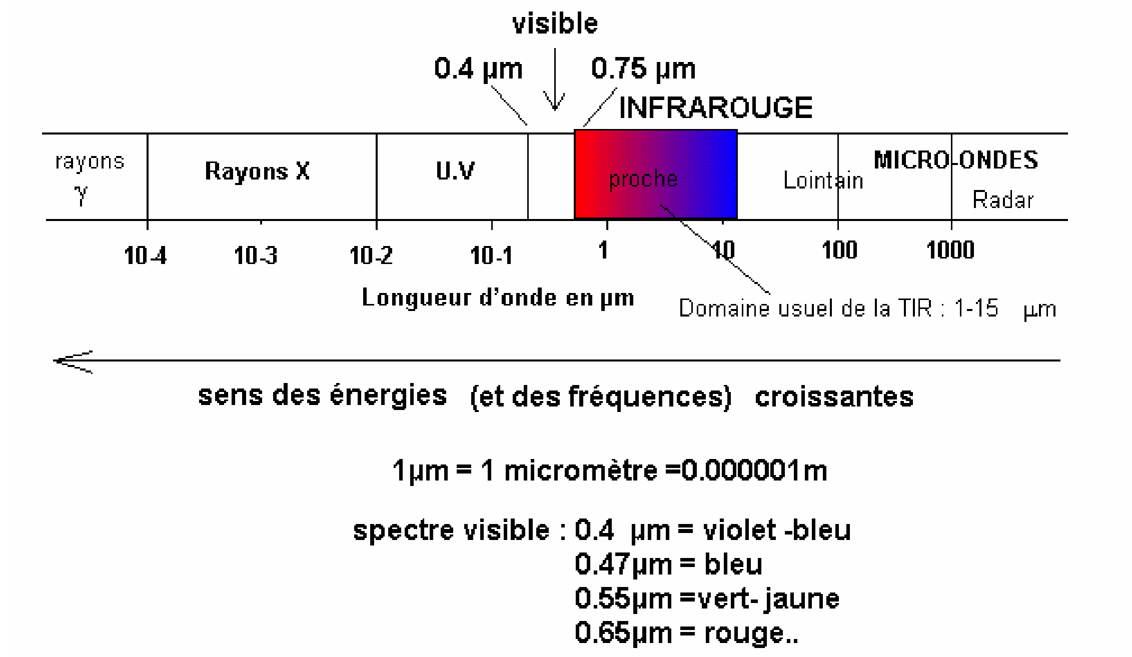
Ex. : IR Camera 256 x 256 pixels 140 Hz 8 bits ⇨ ***8.75 MB/s***

Heat transfer parameters estimation



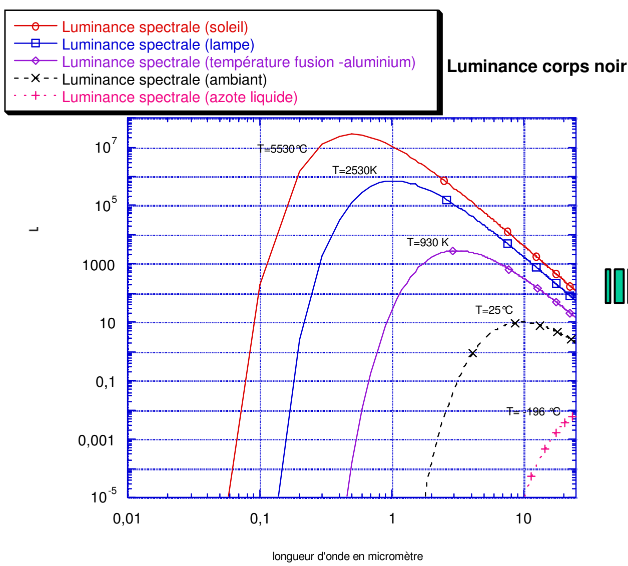
Introduction: IR thermography and Parameters mapping

Thermal Radiation



Infrared thermography : usually 1 – 15 µm

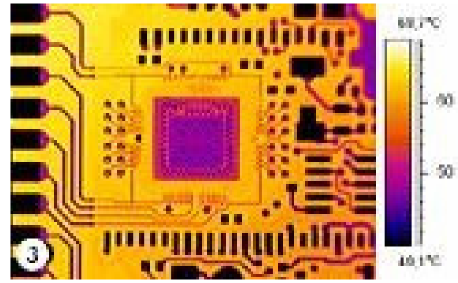
Black body spectral radiance



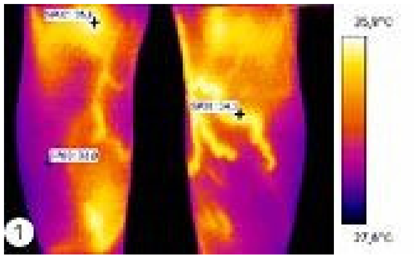
Energy ➔ Signal / Noise

Sensitivity ➔ Resolution

Infrared thermography

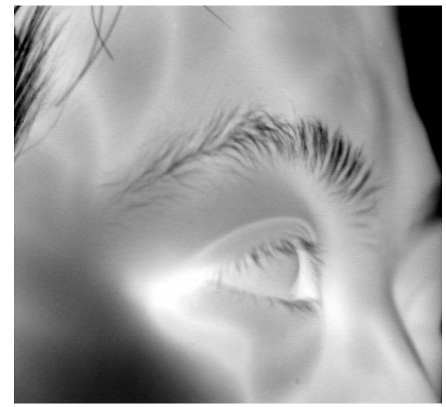


Microcomponent analysis

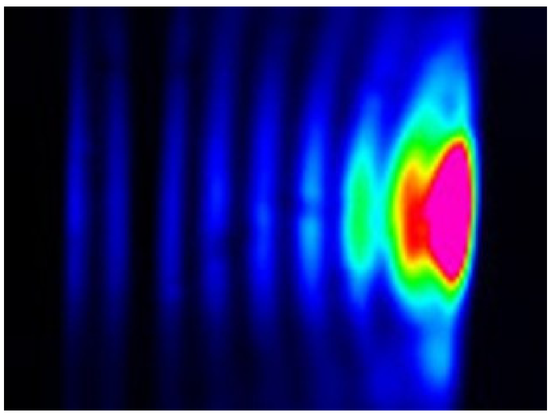


Vascular

Qualitative



Skin & Fever



Microwave heating

Art restoration

Quantitative

Temperature measurements

Calibration
Emissivity
Radiometric equations

Relative temperature measurements
Or even only propotional…

- **Sensor : InSb, InGaAs, MCT, microbolometric…**
- **Focal plane Array: 640x512 pixels or 320x256 pixels**
- **Typical : 150 – 400 Hz !!!**
- **thermal sensibility 30 °C: < 20 mK InSb, MCT**

  < 85 mK μbolometric
- **Spectral Sens. : 1 - 5 μm or 3 - 5 μm (InSb), 8 - 12 μm**
- **Integration time: about 10 μs**
- **One pixel Sensor = 30 μm**

Velocity and heat transfer parameters mapping:

infrared image processing

1. Introduction: IR thermography and Parameters mapping
2. Field Estimation for Local Mapping
3. Macroscopic characterization from averaging
4. Modal approach (SVD)

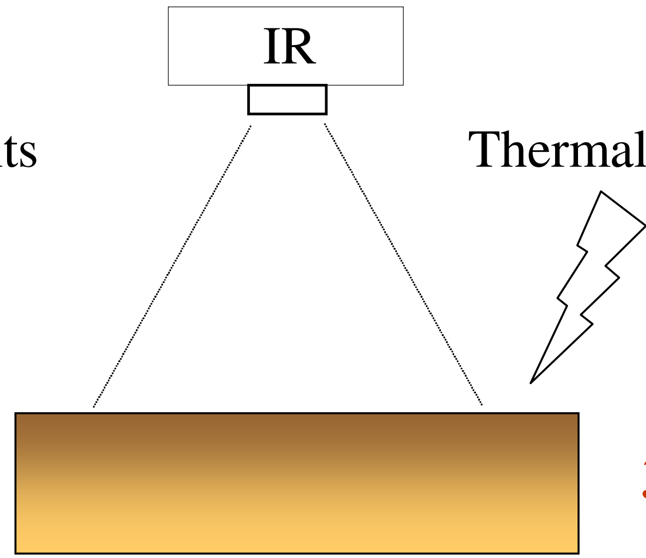
Our specific problem: IR Imaging and Heat Transfer

2D Signal – 2D Estimation – BUT: 3D equations

Thermal Surface Measurements

Thermal Excitation

Self-emission phenomenon
And
Variations in emissivity



3D Heat transfer

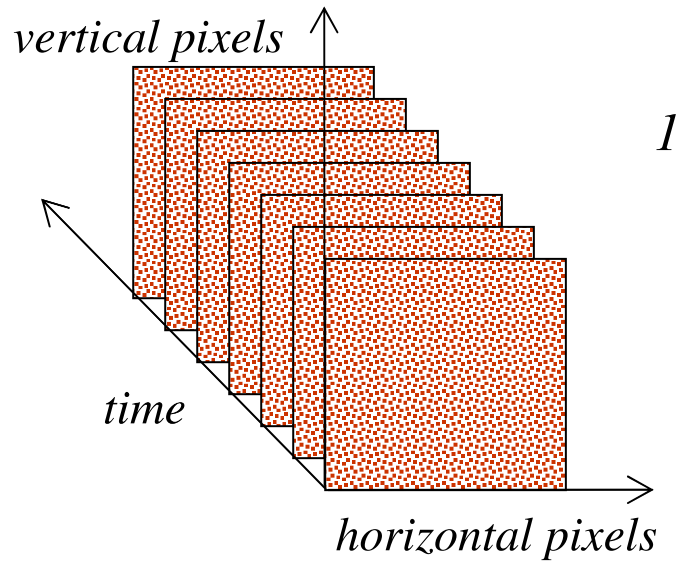
$$\frac{\partial T}{\partial t}+V_x\frac{\partial T}{\partial x}+V_y\frac{\partial T}{\partial y}=\frac{1}{\rho c}\left(\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right)$$

Processing ??? Reduction ??? 3D to 2D ???

IR Imaging and Heat Transfer: Estimation strategy

Reduction of the cube of information
Modal or nodal approaches ???

modal = orthogonal transforms
nodal = discretization



320x256 px
14 bits ; 100 Hz

15 MB/s

Flaw depth in composite structures, delamination, etc…

Global processing
BUT (due to 3D heat transfer)
Difficult to link singular vectors to parameters (qualitative)

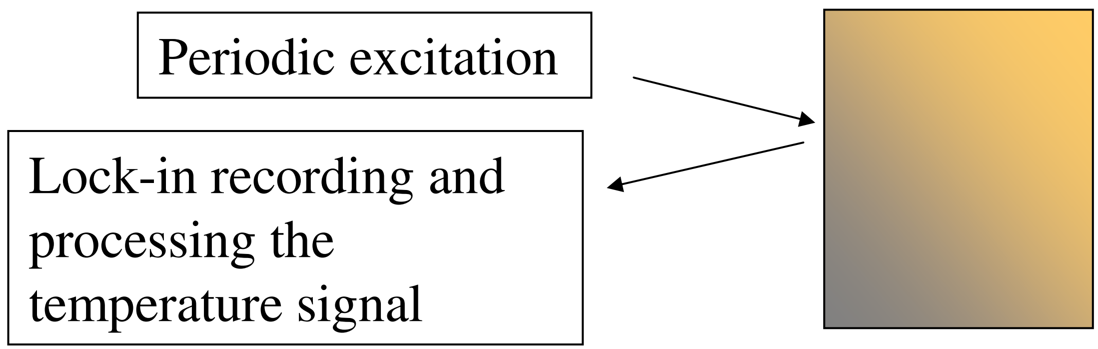
X. Maldague
Theory and practice of infrared technology for Nondestructive Testing, John Wiley, 2001

N. Rajic, 2002, Composite structures

IR Imaging and Heat Transfer: Estimation strategy

Local 1D in-depth transfer (pixels non spatially correlated)

Periodic excitation and Fourier transform / time



Depth range inversely proportional to Modulation frequencies

Higher modulation
=
Near-surface region

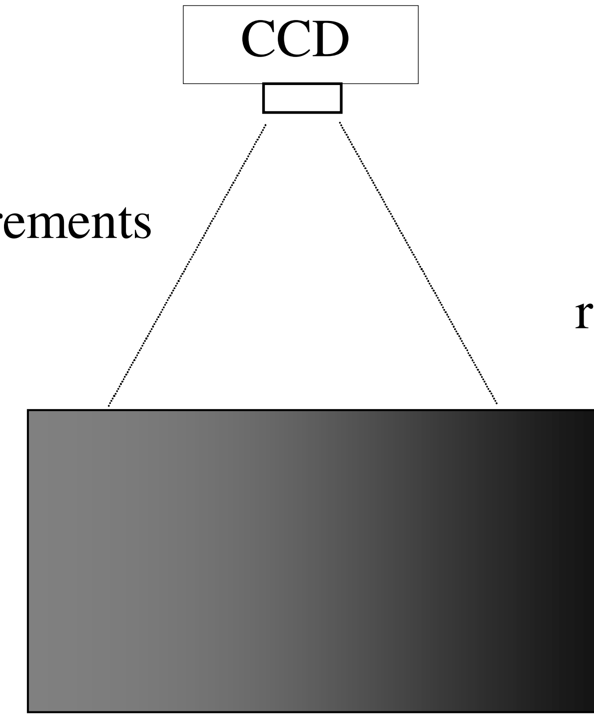
Phase or **Magnitude** image analysis

Not always convenient: small scales, short transient times, chemical reactions…

Wu D., Wu C. Y., Busse G.,
Investigation of resolution in lock-in thermography: Theory and experiment,
Eurotherm Quantitative Infrared Thermography QIRT'96

Computer Vision: Motion Analysis

Optical flow methods



Light Intensity Measurements

Reflection

And

reflectibity differences

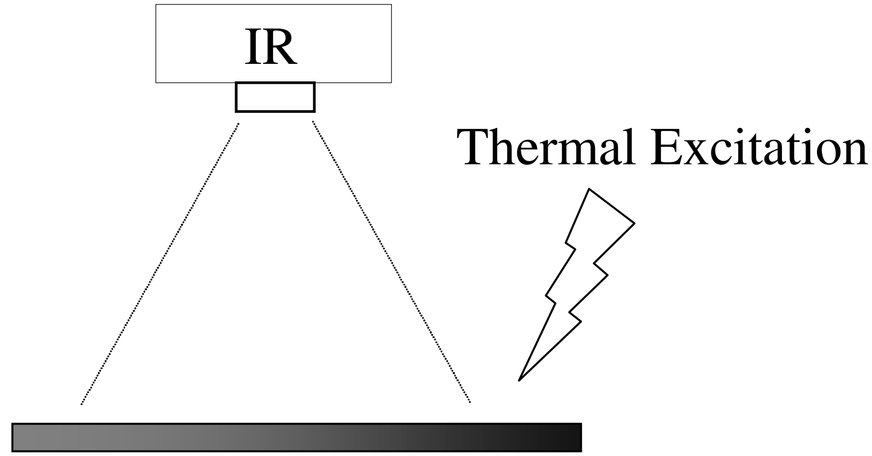
Brightness constancy constraint equation (BCCE)

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial t} = 0$$

⇨ **2D Signal – 2D Estimation – 2D Equations**

Jähne B.
Digital image processing-concepts, algorithms, and scientific application, Springer, Berlin, 1991

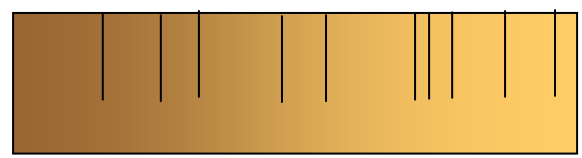
Reduction to 2D Signal analysis



1/ Thin plate / in-plane transfer

2/ Separable solution in thick medium

Vertical cracks

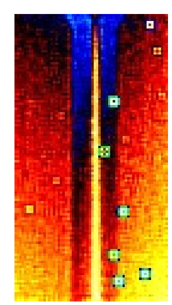


Moving homogeneous solid

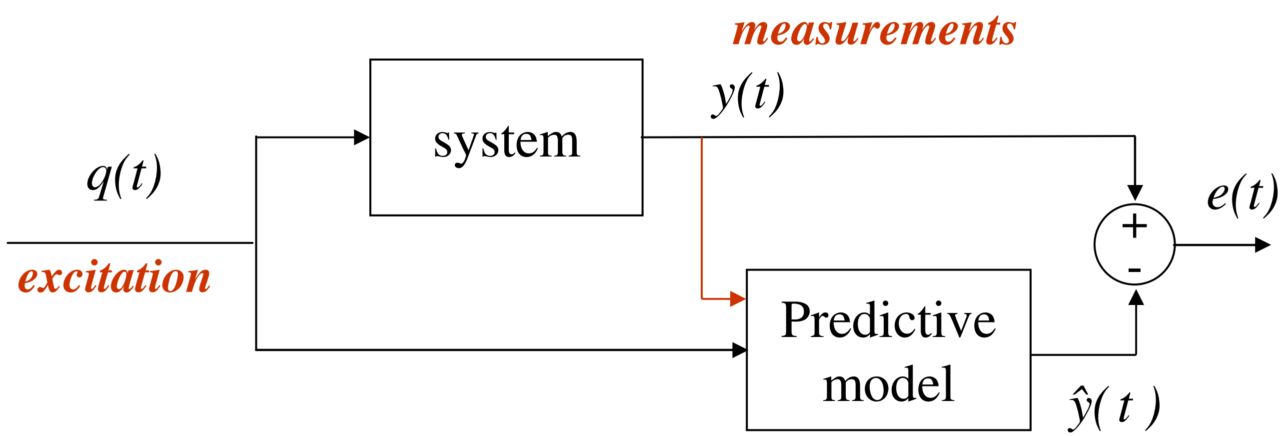
Microfluidic chip

3/ Macroscopic approach from averaging

Fin model in microfluidic chip



Linear estimation : minimization of the prediction error *e(t)*



The regression matrix is filled with measurements

Sampled system $$y(t_k) = \mathbf{H}(t_k)\boldsymbol{\beta} + e(t_k), \qquad \hat{y}(t_k) = \mathbf{H}(t_k)\boldsymbol{\beta}$$

n successive measurements $$\mathbf{Y}_n = \mathbf{H}_n\boldsymbol{\beta} + \mathbf{E}_n$$

OLS Estimation $$\hat{\boldsymbol{\beta}} = \left(\mathbf{H}_n^t\mathbf{H}_n\right)^{-1}\mathbf{H}_n^t\mathbf{Y}_n$$

Linear estimation : minimization of the prediction error *e(t)*

The cost = biased estimador

$$E(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta} + \left(E(\mathbf{H}_n^t \mathbf{H}_n)\right)^{-1} E(\mathbf{H}_n^t \mathbf{E}_n)$$

Bias is zero if :

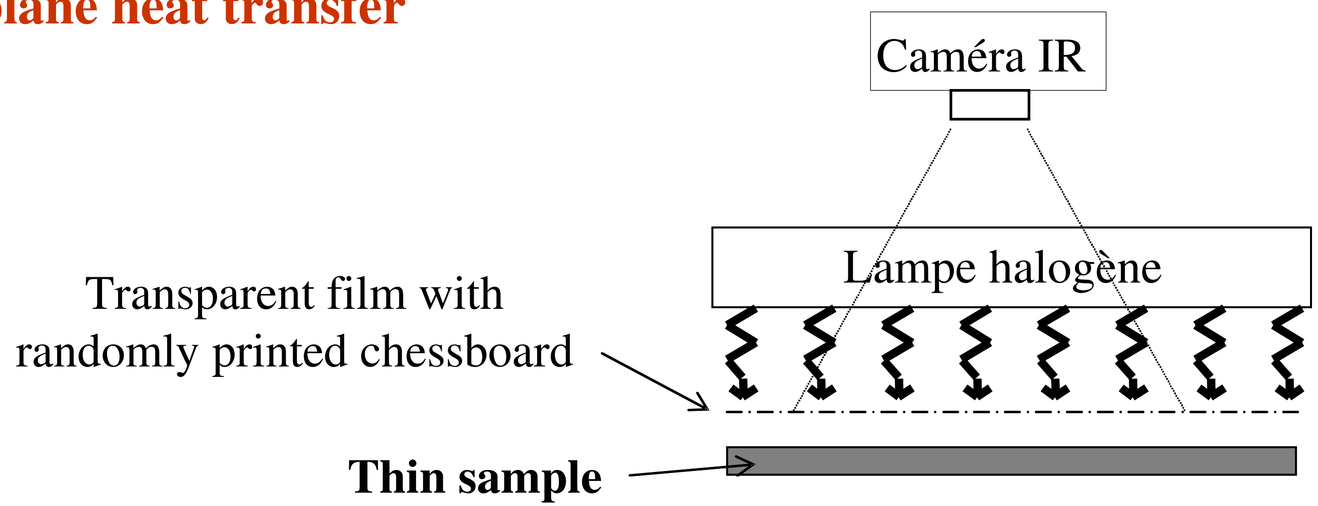
$E(\mathbf{H}_n^t \mathbf{H}_n)$ non singular & $e(t)$ is a white noise

or

The input $q(t)$ is independant of $e(t)$ and $H(t)$ does not depend of $y(t)$

Thermal diffusivity mapping from spatial random heating

2D In-plane heat transfer



$$\rho c(x,y)\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(k(x,y)\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k(x,y)\frac{\partial T}{\partial y}\right)-\frac{2h}{e}(T-T_\infty)$$

Discretization

$$\mathbf{T}^{\mathbf{t}+\Delta t}-\mathbf{T}^{\mathbf{t}}=\mathbf{A}.*\Delta\mathbf{T}^{\mathbf{t}}+\mathbf{C}^{-1}.*\delta_{\mathbf{x}}\mathbf{K}.*\delta_{\mathbf{x}}\mathbf{T}^{\mathbf{t}}+\mathbf{C}^{-1}.*\delta_{y}\mathbf{K}.*\delta_{y}\mathbf{T}^{\mathbf{t}}-\mathbf{H}.*(\mathbf{T}^{\mathbf{t}}-T_\infty)$$

Linear least squares
Maximum likelihood Estimator

$$\mathbf{T} = \mathbf{X\beta}$$

Hypothesis :

- zero mean and additive errors

- β constant and unknown before the estimation and X_{ij} known without error

- constant variance (σ known) and uncorrelated errors

$$S = (\mathbf{Y} - \mathbf{X\beta})^t . (\mathbf{Y} - \mathbf{X\beta})$$

Estimator $$\hat{\boldsymbol{\beta}} = (\mathbf{X}^t\mathbf{X})^{-1}.\mathbf{X}^t\mathbf{Y}$$

Estimation error $$cov(\mathbf{e}_{\boldsymbol{\beta}}) = (\mathbf{X}^t\mathbf{X})^{-1}.\sigma^2$$

Thermal diffusivity mapping from spatial random heating

$$\hat{\mathbf{T}}' - \hat{\mathbf{T}} = \begin{bmatrix} \hat{\mathbf{T}}^{\mathbf{t}_0+\Delta\mathbf{t}} - \hat{\mathbf{T}}^{\mathbf{t}_0} \\ . \\ \hat{\mathbf{T}}^{\mathbf{t}+\Delta\mathbf{t}} - \hat{\mathbf{T}}^{\mathbf{t}} \\ . \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} \Delta\hat{\mathbf{T}}^{\mathbf{t}_0} & \delta_x \hat{\mathbf{T}}^{\mathbf{t}_0} & \delta_y \hat{\mathbf{T}}^{\mathbf{t}_0} & \hat{\mathbf{T}}^{\mathbf{t}_0} - T_\infty \\ . & . & . & . \\ \Delta\hat{\mathbf{T}}^{\mathbf{t}} & \delta_x \hat{\mathbf{T}}^{\mathbf{t}} & \delta_y \hat{\mathbf{T}}^{\mathbf{t}} & \hat{\mathbf{T}}^{\mathbf{t}} - T_\infty \\ . & . & . & . \end{bmatrix}$$

$$\hat{\beta} = \left(\mathbf{X}^t \mathbf{X}\right)^{-1} \mathbf{X}^t \left(\hat{\mathbf{T}}' - \hat{\mathbf{T}}\right)$$

Point by point estimation

$\mathbf{X}^t\mathbf{X}$ = 4x4 matrix

$$\beta_{ij} = \begin{bmatrix} a_{ij} \\ \dfrac{\delta_x k_{ij}}{(\rho c)_{ij}} \\ \dfrac{\delta_y k_{ij}}{(\rho c)_{ij}} \\ H_{ij} \end{bmatrix}$$

Sequential implementation of the sums
(Recursive estrimation)

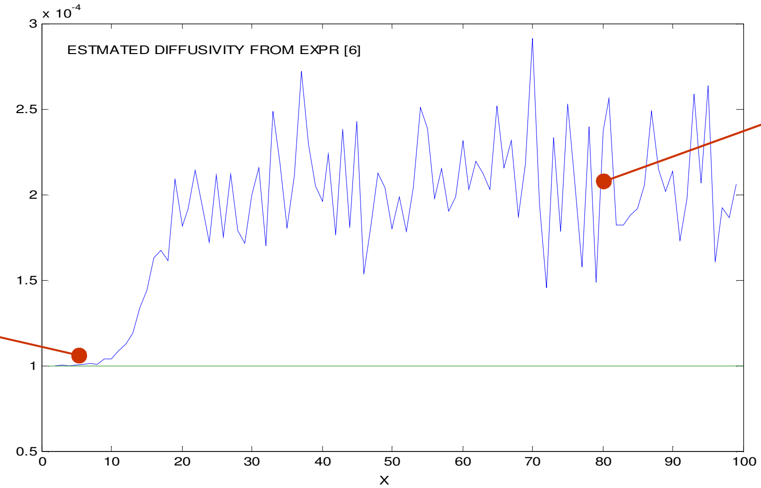
Simplified model

$$\hat{\beta} \equiv \mathbf{A} \Rightarrow \hat{\mathbf{A}} = \frac{\sum_{i=0}^{n} \Delta\hat{\mathbf{T}}^{t_i} .* \left(\hat{\mathbf{T}}^{t_i+\Delta t} - \hat{\mathbf{T}}^{t_i}\right)}{\sum_{i=0}^{n} \left(\Delta\hat{\mathbf{T}}^{t_i}\right)^2}$$

Thermal diffusivity mapping from spatial random heating

Correct estimation in the perturbated region

Important bias in the unperturbated region



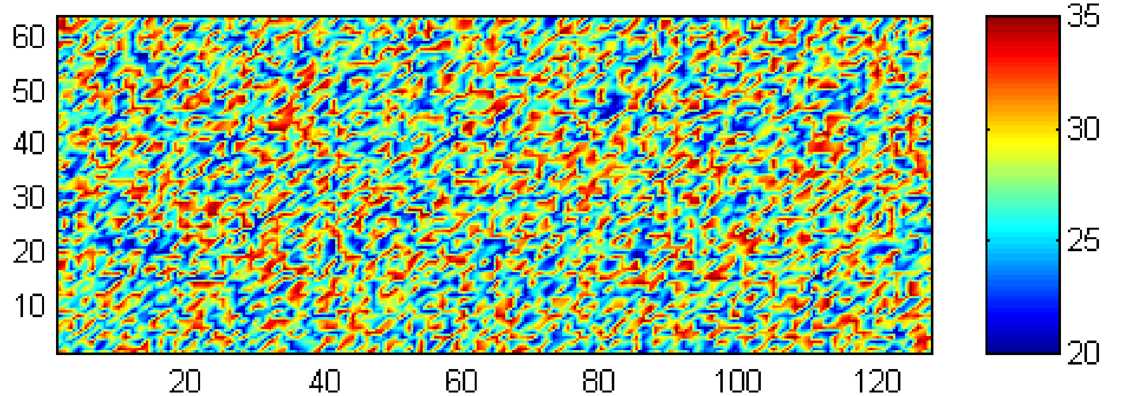
Homogeneous plate with local heating

Periodic heating ⇒ Periodic bias in the unperturbated points

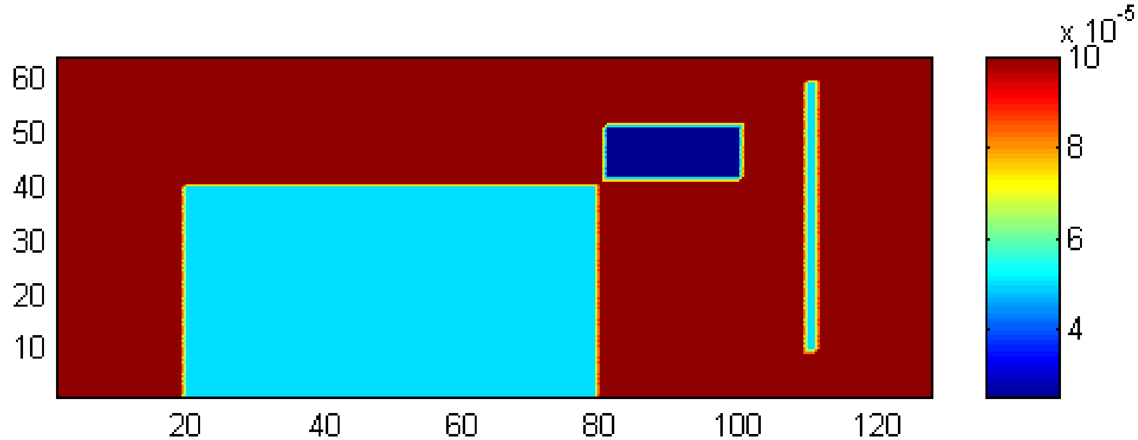
$$\hat{\mathbf{A}} = \frac{\sum_{i=0}^{n} \mathbf{\Delta}\hat{\mathbf{T}}^{t_i} .* \left(\hat{\mathbf{T}}^{t_i+\Delta t} - \hat{\mathbf{T}}^{t_i}\right)}{\sum_{i=0}^{n} \left(\mathbf{\Delta}\hat{\mathbf{T}}^{t_i}\right)^2}$$

Thermal diffusivity mapping from spatial random heating

Initial randomly distributed temperature field

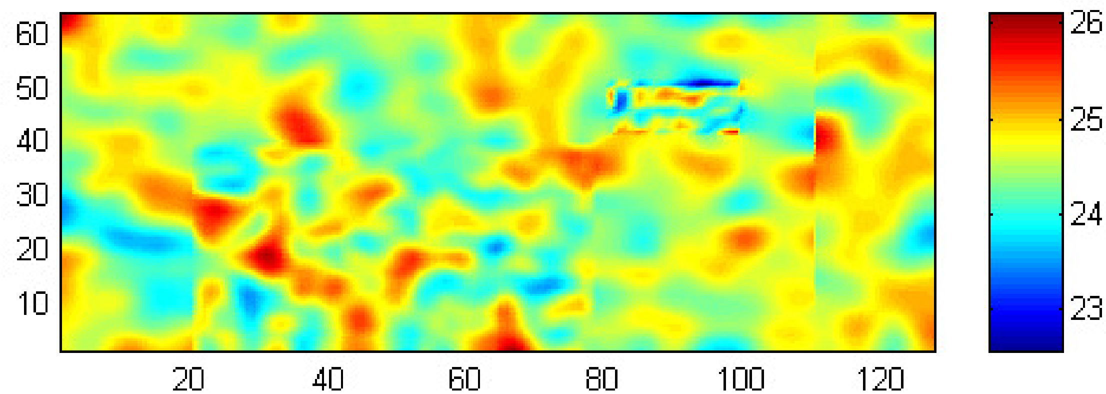


Sample

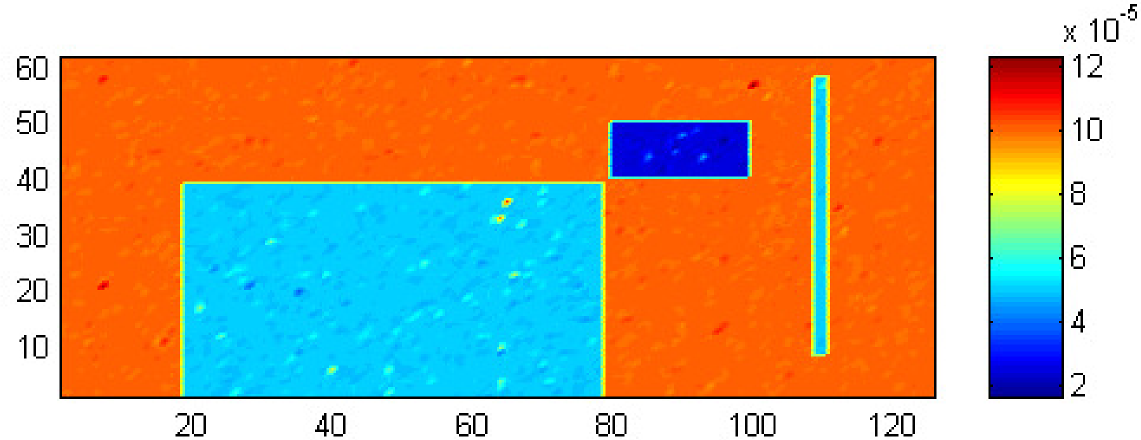


Thermal diffusivity mapping from spatial random heating

Final temperature field



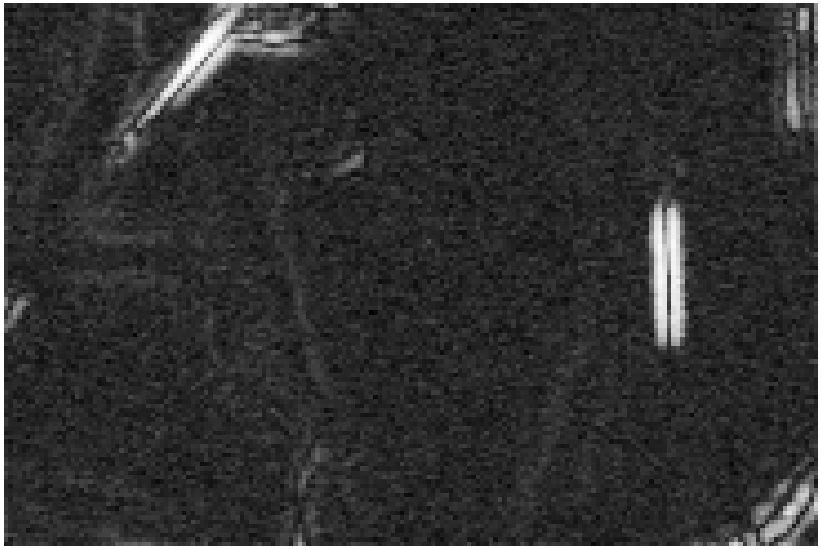
Estimated thermal diffusivity field



Experimental Results : crack detection on an aluminium sheet

Initial thermal field

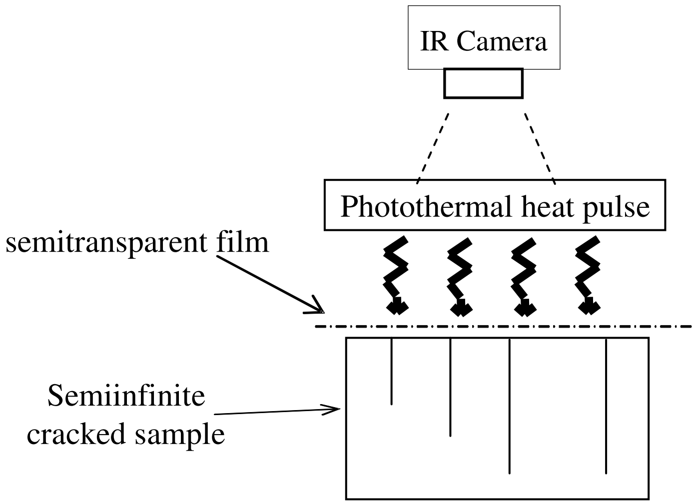
Thermal diffusivity mapping



Batsale JC, Battaglia JL, Fudym O.

*Autoregresseive algorithms and spatially random flash excitation for 2D non-destructive evaluation with IR cameras, QIRT Journal **1** 1-20, 2004.*

Thermal diffusivity mapping from spatial random heating



Thick Sample
(semi-infinite)

$$T_z(0,t) = C / \sqrt{t}$$

$$T(x,y,z,t) = T_{x,y}(x,y,t).T_z(z,t)$$

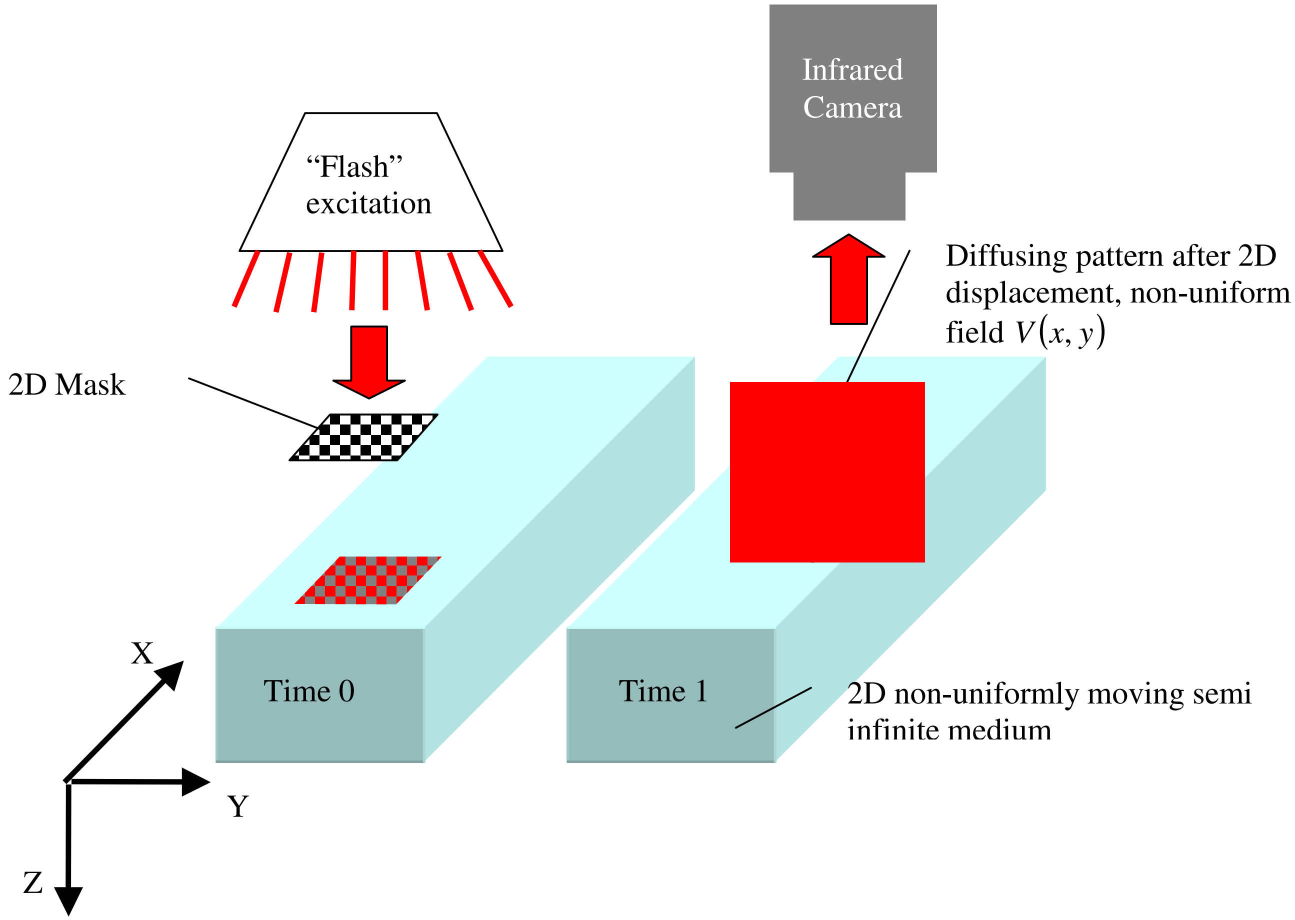
Separability
(from Quadrupole approach)

$$T_{x,y}(x,y,t) = T(x,y,z=0,t).\sqrt{t}$$

New Observable variable

$$\hat{\boldsymbol{\beta}}_{ML} = \left(\hat{\mathbf{X}}'\mathbf{t}^{-1}\hat{\mathbf{X}}\right)^{-1}\hat{\mathbf{X}}'\mathbf{t}^{-1}\left(\hat{\mathbf{Y}}^{\mathbf{t+\Delta t}} - \hat{\mathbf{Y}}^{\mathbf{t}}\right)$$

Velocity and diffusion mapping for a moving solid



$$\frac{\partial T(x,y,z,t)}{\partial t}+V_x\frac{\partial T(x,y,z,t)}{\partial x}+V_y\frac{\partial T(x,y,z,t)}{\partial y}=a\cdot\left(\frac{\partial^2 T(x,y,z,t)}{\partial x^2}+\frac{\partial^2 T(x,y,z,t)}{\partial y^2}+\frac{\partial^2 T(x,y,z,t)}{\partial z^2}\right)$$

Total Least Square Estimation

$$\frac{\partial T_{x,y}(x,y,t)}{\partial t} + V_x \frac{\partial T_{x,y}(x,y,t)}{\partial x} + V_y \frac{\partial T_{x,y}(x,y,t)}{\partial y} = a \cdot \left(\frac{\partial^2 T_{x,y}(x,y,t)}{\partial x^2} + \frac{\partial^2 T_{x,y}(x,y,t)}{\partial y^2} \right)$$

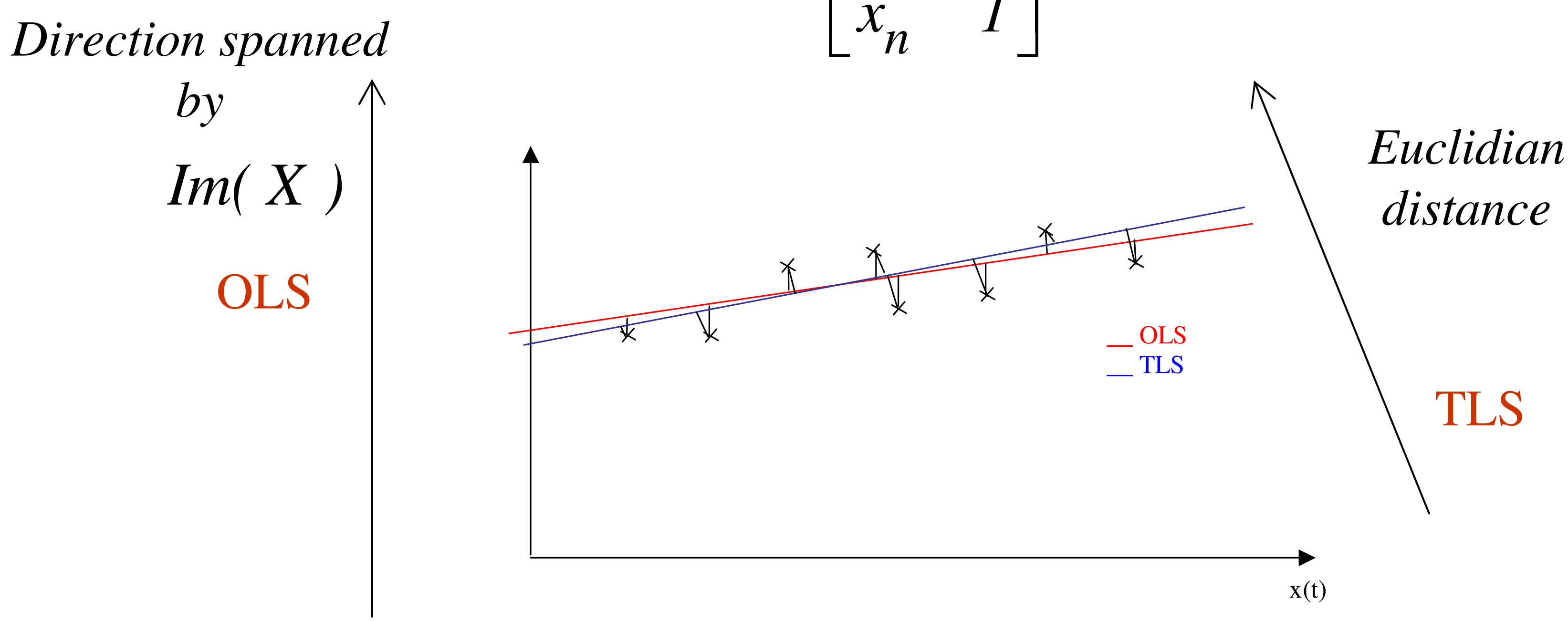
$$X\boldsymbol{\beta} = \mathbf{Y} \quad \Longrightarrow \quad \begin{bmatrix} \mathbf{X} & \mathbf{-Y} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta} \\ 1 \end{bmatrix} = \mathbf{0}$$

$$D(x,y) \quad (x,y) = 0$$

$$\begin{cases} (x,y) = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \frac{\partial T}{\partial x}(x,y,t_i) & \frac{\partial T}{\partial y}(x,y,t_i) & -\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)(x,y,t) & \frac{\partial T}{\partial t}(x,y,t_i) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \\ (x,y) = \begin{bmatrix} V_x(x,y) & V_y(x,y) & a(x,y) & 1 \end{bmatrix}^T \end{cases}$$

Example for linear regression

$$X = \begin{bmatrix} x_1 & 1 \\ \dots & \dots \\ x_n & 1 \end{bmatrix}$$



Total Least Square Estimation

$$\left\{ \quad (x, y), \ \| \quad (x, y) \quad (x, y) \|^{2} = min \right\}$$

With the constraint $\| \quad (x, y) \|^{2} = 1$

$$\left\{ \quad (x, y), \ \| \quad (x, y) \quad (x, y) \|^{2} + \lambda (x, y) \left(1 - \| \quad (x, y) \|^{2} \right) = min \right\}$$

Lagrange multipiers

$$(x, y) = \quad (x, y)^{T} \quad (x, y)$$

Total Least Square Estimation

Minimum for $\left(x,y\right)\quad\left(x,y\right)=\lambda\left(x,y\right)\quad\left(x,y\right)$

$$V_{min}\left(x,y\right) \qquad\qquad \lambda_{min}\left(x,y\right)$$

Eigenvector associated with the minimum eigenvalue

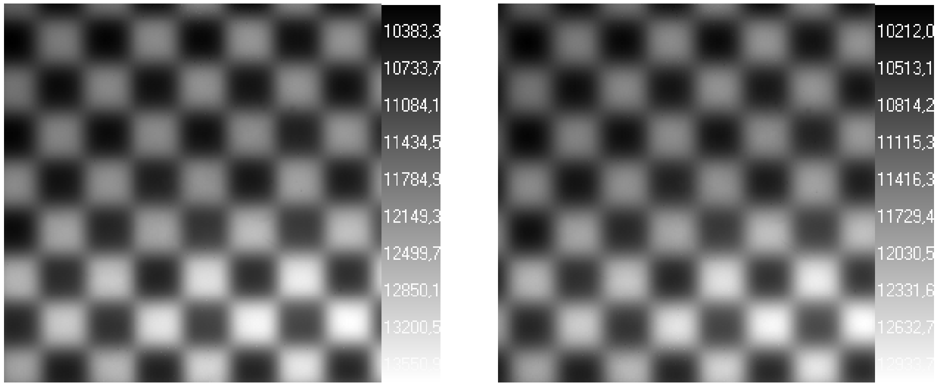
BUT $\lambda_N \geq \lambda_{N-1} \geq .. > \lambda_p \approx .. \approx \lambda_0 \approx 0$

Threshold ?

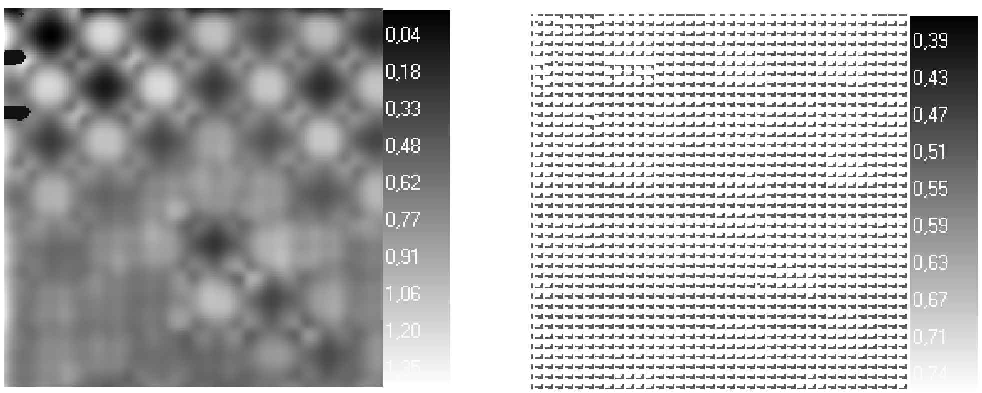
Noise subspace dimension ?

- spanned by the eigenvectors of the “close to zero” eigenvalues -

Velocity and diffusion mapping for a moving solid



Infrared sequence showing a moving and diffusing pattern

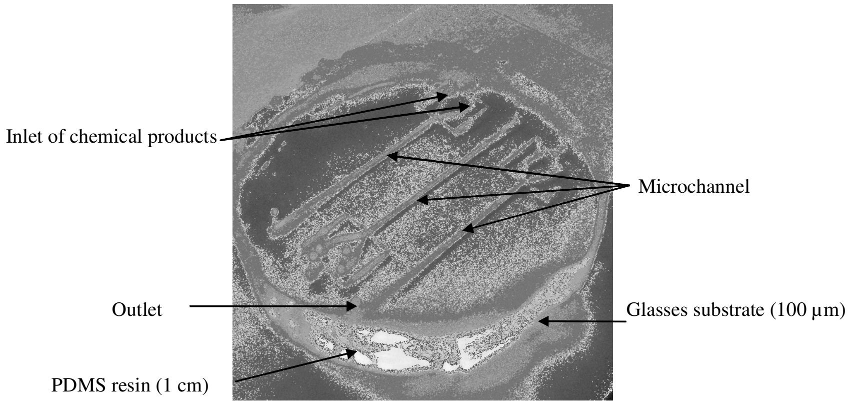


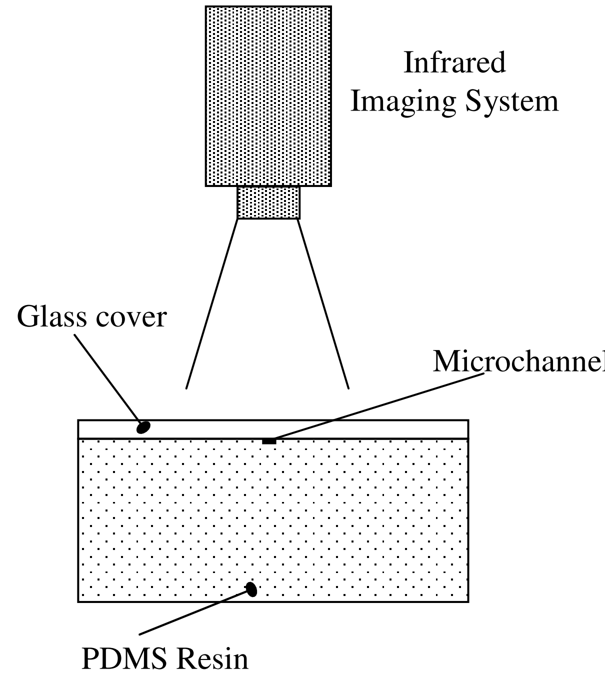
Diffusivity and velocity mapping from previous image sequence sampled at 25 Hz

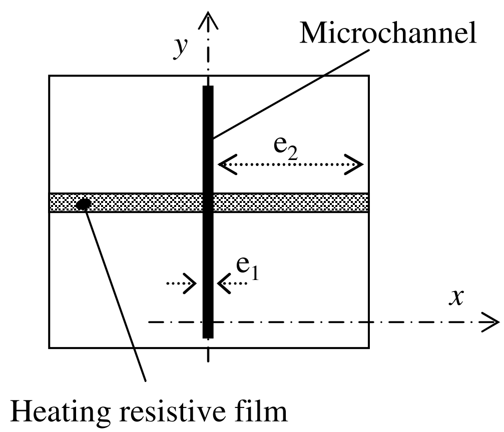
Velocity and heat transfer parameters mapping: infrared image processing

1. Introduction: IR thermography and Parameters mapping
2. Field Estimation for Local Mapping
3. Macroscopic characterization from averaging
4. Modal approach (SVD)

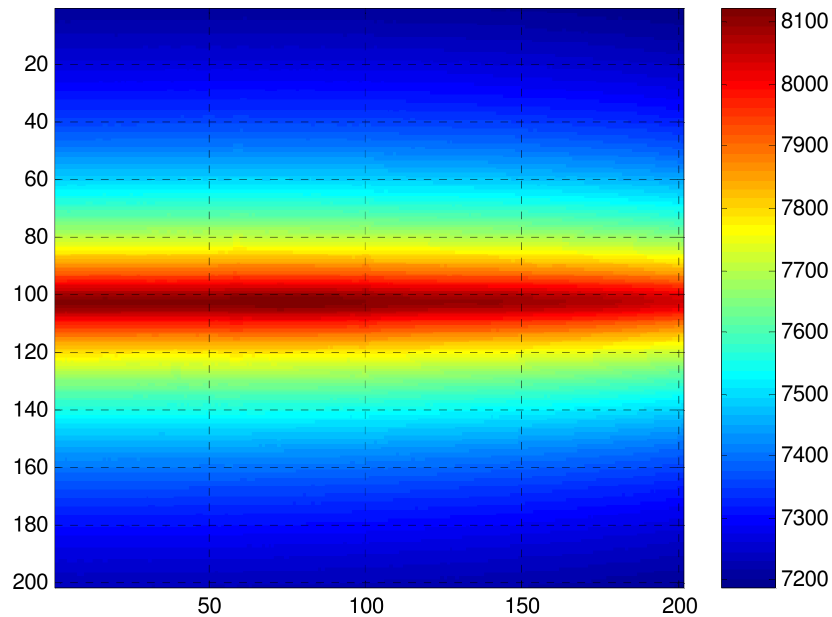
Macroscopic characterization from averaging

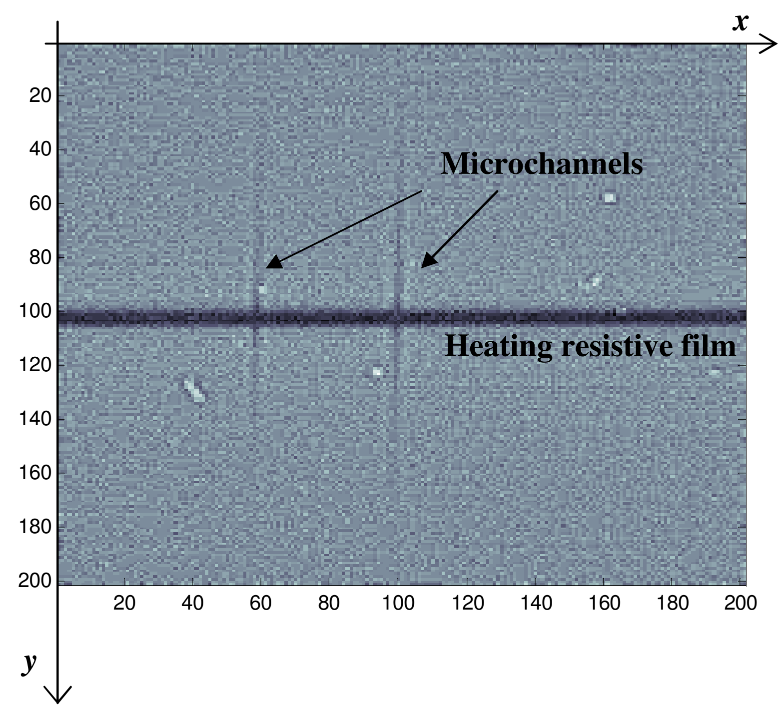






Macroscopic characterization from averaging





$$\frac{\partial}{\partial x}\left(k\frac{\partial Y}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial Y}{\partial y}\right)+\frac{g}{k}-HY=0$$

$$\Delta Y=-\frac{\partial k}{\partial x}\frac{\partial Y}{\partial x}-\frac{\partial k}{\partial y}\frac{\partial Y}{\partial y}-\frac{g}{k}+HY$$

Macroscopic characterization from averaging

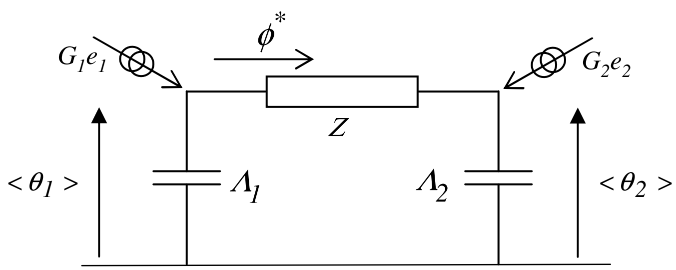
$$\frac{\partial^2 T_i}{\partial x^2} + \frac{\partial^2 T_i}{\partial y^2} + \frac{g_i(y,t)}{k_i} = \frac{1}{a_i}\frac{\partial T_i}{\partial t} + \frac{v_i}{a_i}\frac{\partial T_i}{\partial y}$$

Analytical Averaged Temperature Model :

$$k_1 \beta_1^2 e_1 < \theta_1 > = G_1 e_1 - \phi^*$$

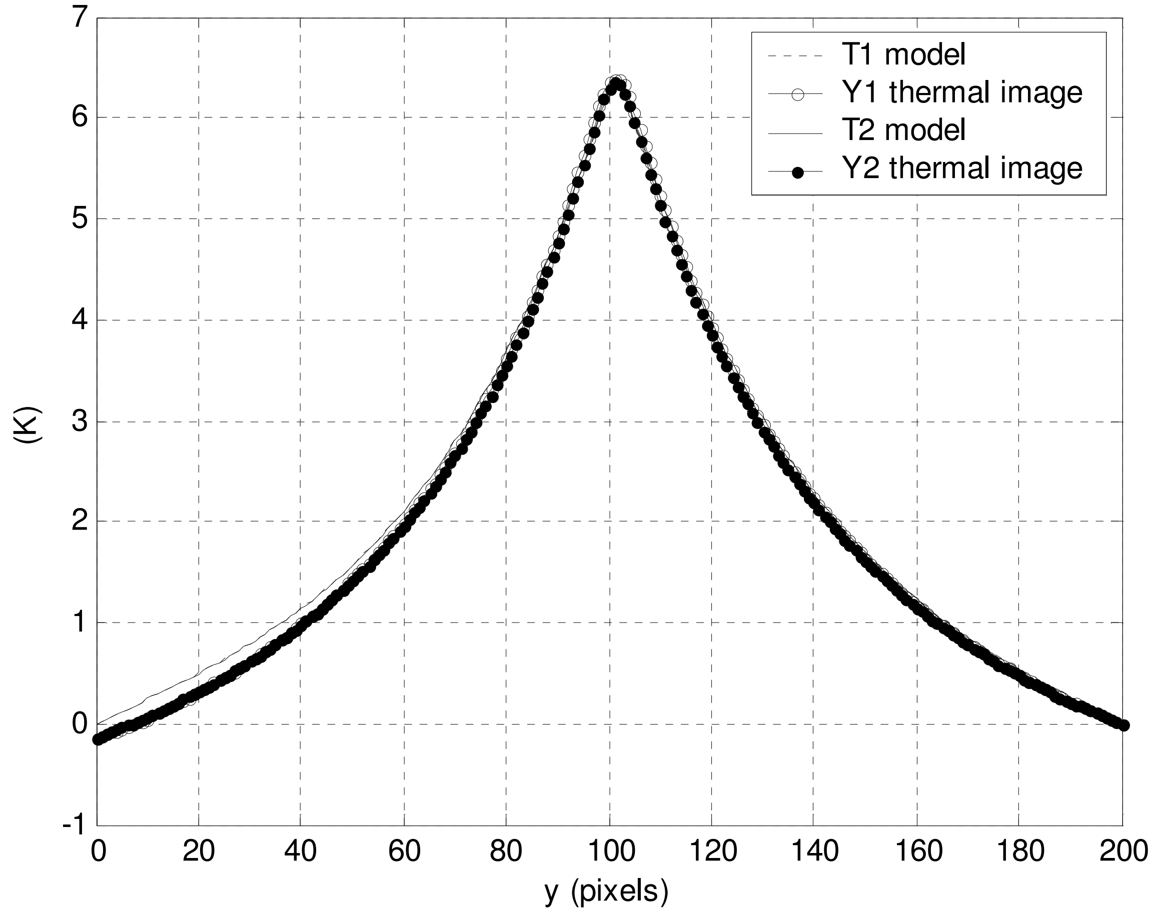
$$k_2 \beta_2^2 e_2 < \theta_2 > = G_2 e_2 + \phi^*$$

$$< \theta_1 > - < \theta_2 > = Z\phi^*$$

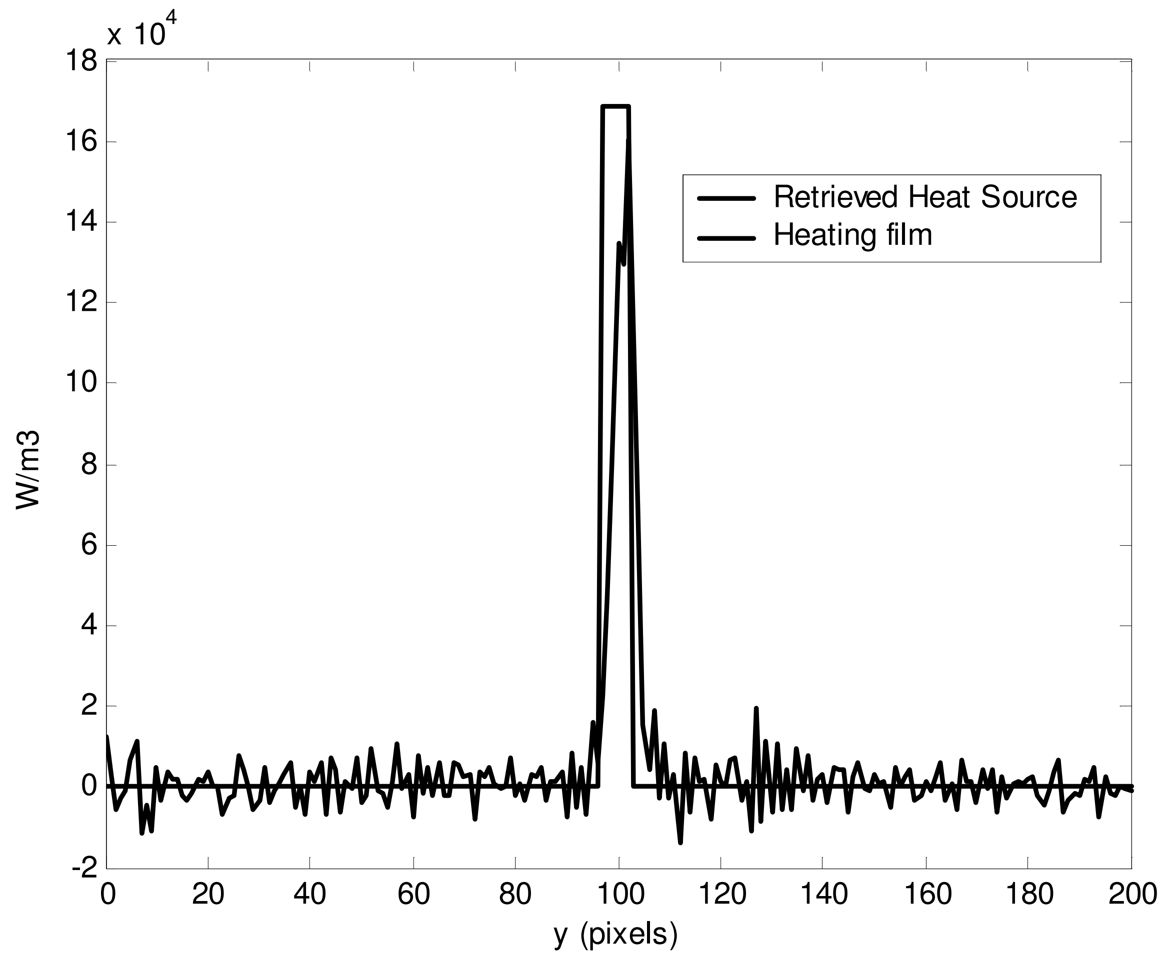


Macroscopic characterization from averaging

Fitted with
Heat losses
H



Macroscopic characterization from averaging



Retrieved Heat Source

Velocity and heat transfer parameters mapping:

infrared image processing

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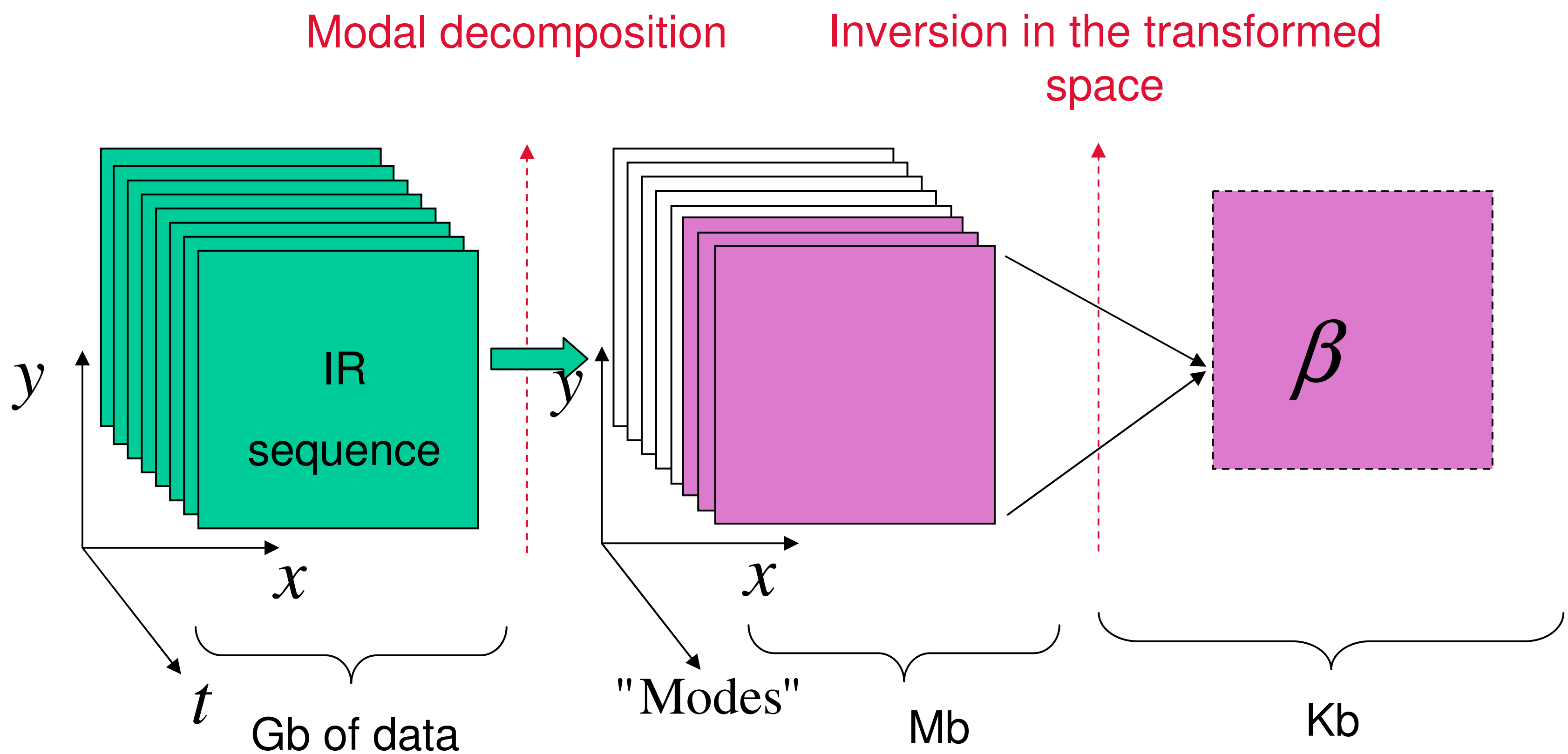
Modal approach

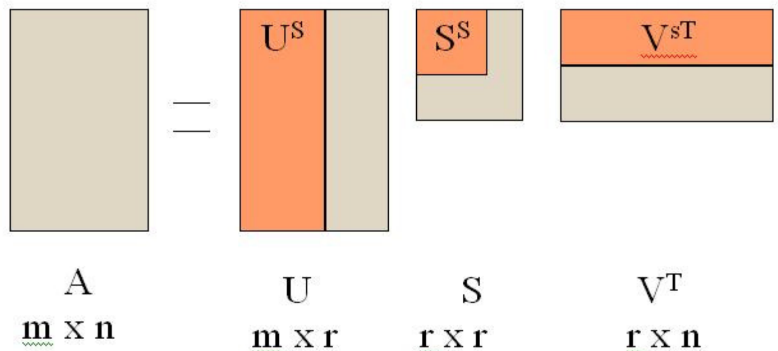


Why Singular Value Decomposition ?

SVD = Projection on

Empirical Orthogonal Functions

According to literature, SVD is very convenient to perform data compression and "denoising" (provided data can be arranged as a 2D matrix)



Compression/denoising

$$A = \sum_{k=1}^{r} \lambda_k \cdot U_k \cdot V_k^{\,T} \approx \sum_{k=1}^{l \ll r} \lambda_k \cdot U_k \cdot V_k^{\,T}$$

Separation properties

$$A(x,t) = \sum_{k=1}^{r} \lambda_k \cdot U_k(x) \cdot V_k(t)^T$$

Equivalent heat equation (1D example)

$$\frac{\partial}{\partial x}\left(a(x)\cdot\frac{\partial T}{\partial x}\right)=\frac{\partial T}{\partial t}$$

⇩

$$a(x)\cdot\frac{\partial^2 U_k(x)}{\partial x^2}+\frac{\partial a(x)}{\partial x}\cdot\frac{\partial U_k(x)}{\partial x}=\sum_{j=1}^{k}U_j(x)\cdot\frac{\lambda_j}{\lambda_k}\cdot\left(\frac{\partial V_j^T}{\partial t}\cdot V_k\right)$$

⇩

$$a(x)\cdot\ddot{U}_k(x)+\frac{\partial a(x)}{\partial x}\dot{U}_k(x)=\tilde{U}_k(x)\qquad k=1..r$$

First term: “Laplacian” 2[d] term: “gradient” 3[rd] term: “observable”

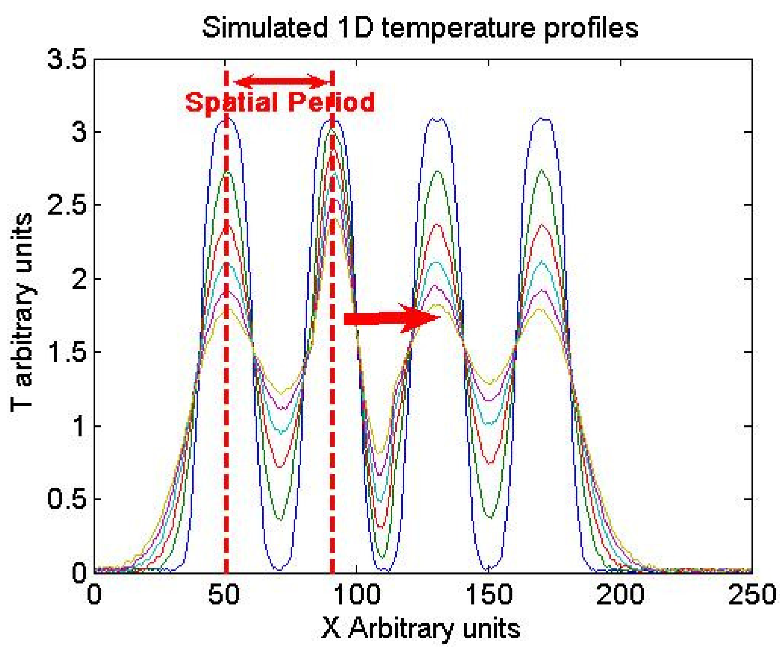
Which modes and which locations are best suited for the estimations of $a(x)$?

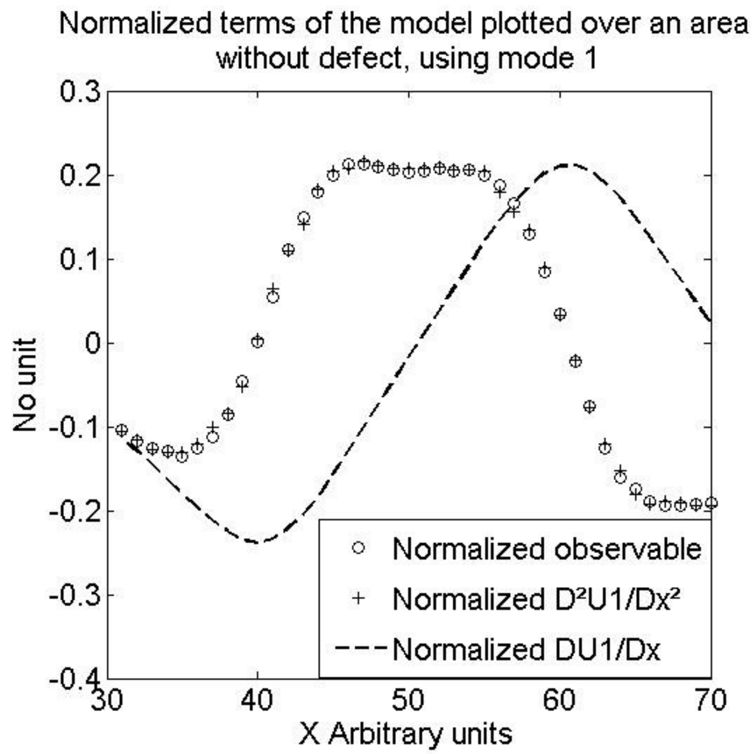
⇩

Need for a sensitivity analysis…

Sensitivity analysis in the transformed space

Locally, we compare the terms of the model

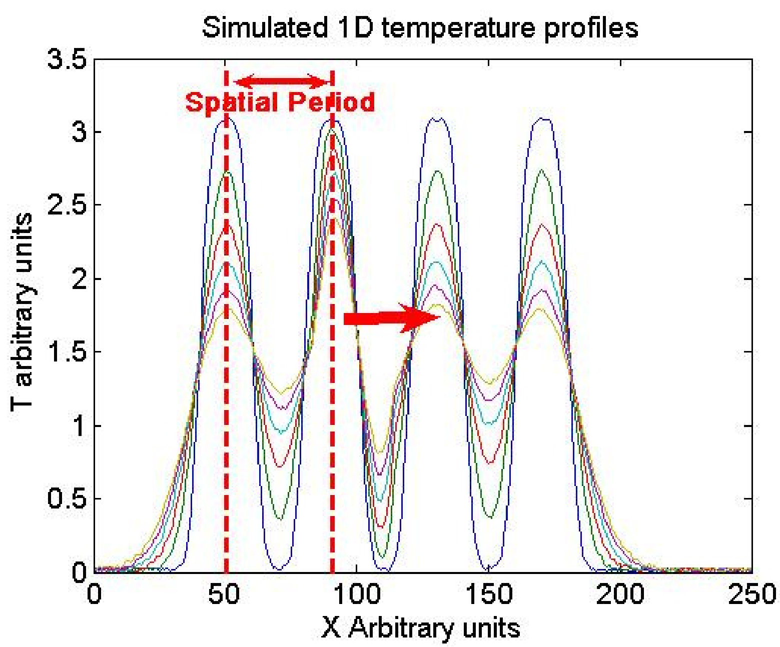


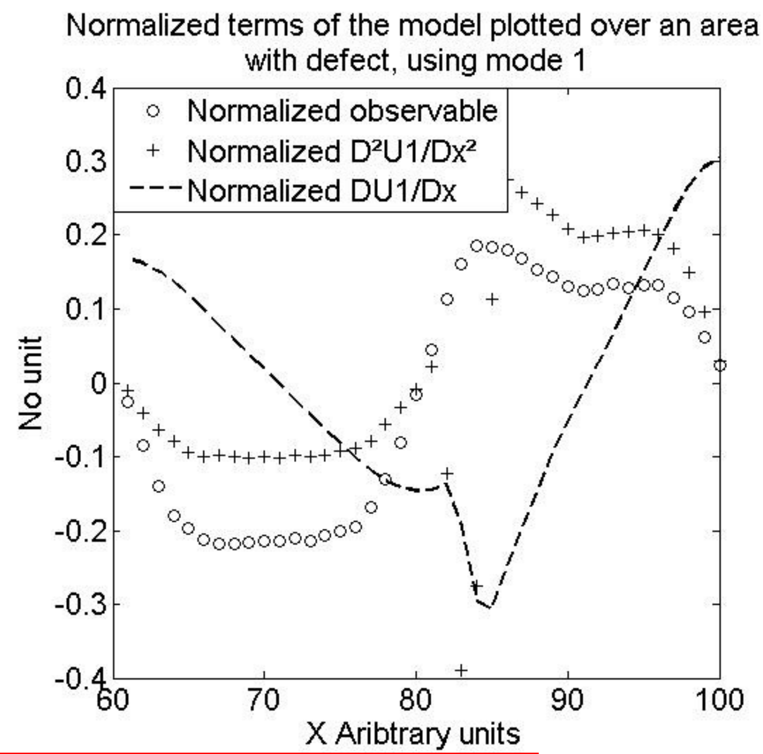


$$\rho\left(\dot{U}_1, \tilde{U}_1\right) = \frac{\left\langle \dot{U}_1 \middle| \tilde{U}_1 \right\rangle}{\left\| \dot{U}_1 \right\| \cdot \left\| \tilde{U}_1 \right\|} \approx 0 \quad \& \quad \left| \rho\left(\ddot{U}_1, \tilde{U}_1\right) \right| = \frac{\left| \left\langle \ddot{U}_1 \middle| \tilde{U}_1 \right\rangle \right|}{\left\| \ddot{U}_1 \right\| \cdot \left\| \tilde{U}_1 \right\|} \approx 1$$

Sensitivity analysis in the transformed space

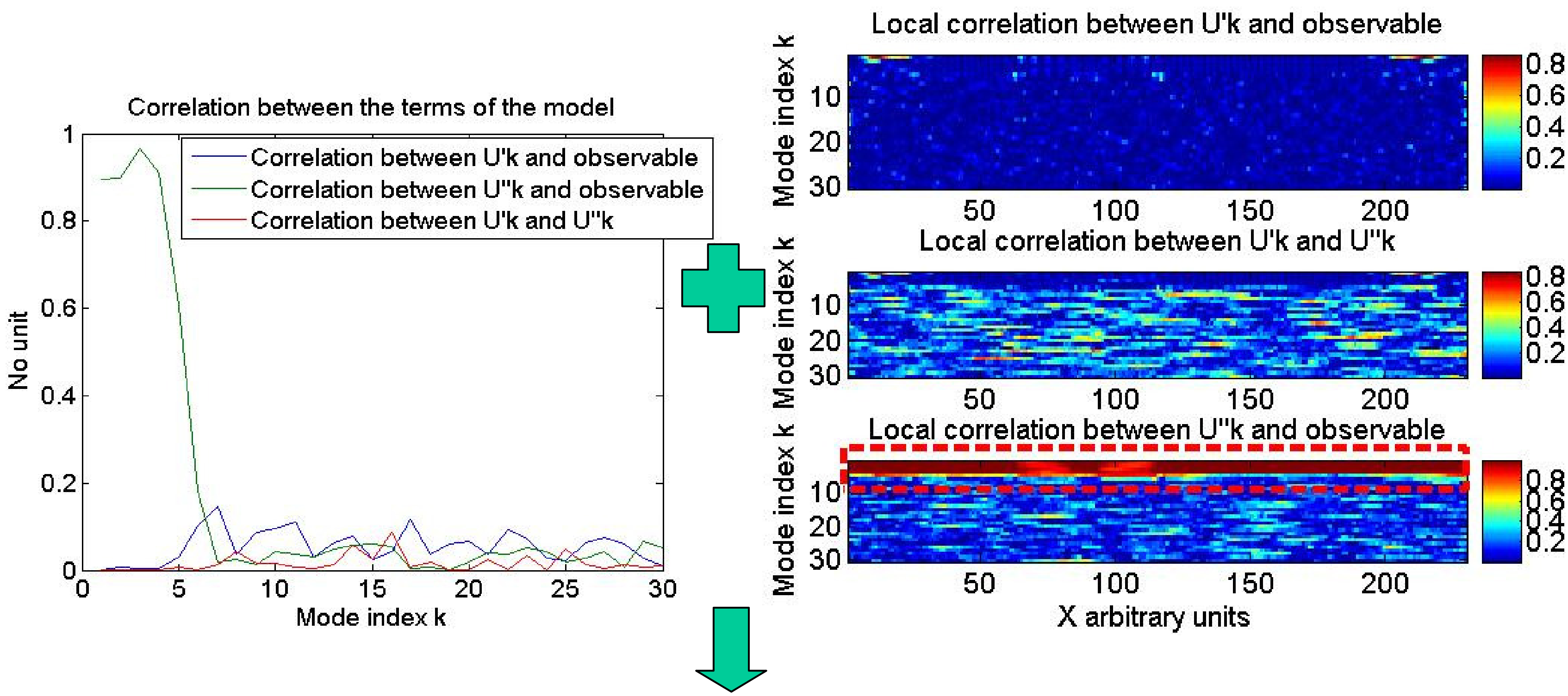
Locally, we compare the terms of the model





$$\left|\rho\left(\dot{U}_1,\tilde{U}_1\right)\right| = \frac{\left|\left\langle \dot{U}_1 \middle| \tilde{U}_1 \right\rangle\right|}{\left\|\dot{U}_1\right\| \cdot \left\|\tilde{U}_1\right\|} > 0 \quad \& \quad \left|\rho\left(\ddot{U}_1,\tilde{U}_1\right)\right| = \frac{\left|\left\langle \ddot{U}_1 \middle| \tilde{U}_1 \right\rangle\right|}{\left\|\ddot{U}_1\right\| \cdot \left\|\tilde{U}_1\right\|} < 1$$

Sensitivity analysis in the transformed space



Physical model can be locally simplified, only the
three first modes of the decomposition are needed

MODEL REDUCTION, DATA COMPRESSION, SENSITIVITY ENHANCEMENT!

From sensitivity analysis to inversion

$k = 1..r$

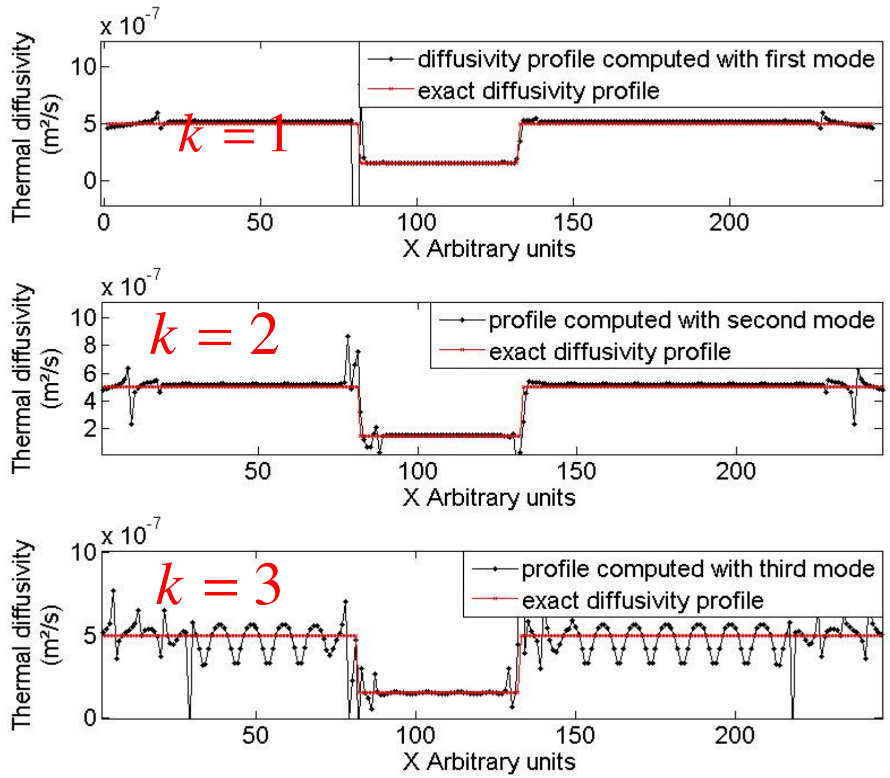
$$a^k(i) = \frac{\rho\left(\ddot{U}_k^i,\tilde{U}_k^i\right) - \rho\left(\dot{U}_k^i,\tilde{U}_k^i\right)\rho\left(\ddot{U}_k^i,\dot{U}_k^i\right)}{1-\rho^2\left(\ddot{U}_k^i,\dot{U}_k^i\right)} \cdot \frac{\left\|\tilde{U}_k^i\right\|}{\left\|\ddot{U}_k^i\right\|}$$

$k = 1..3$

$$a^k(i) = -\rho\left(\ddot{U}_k^i,\tilde{U}_k^i\right) \cdot \frac{\left\|\tilde{U}_k^i\right\|}{\left\|\ddot{U}_k^i\right\|}$$

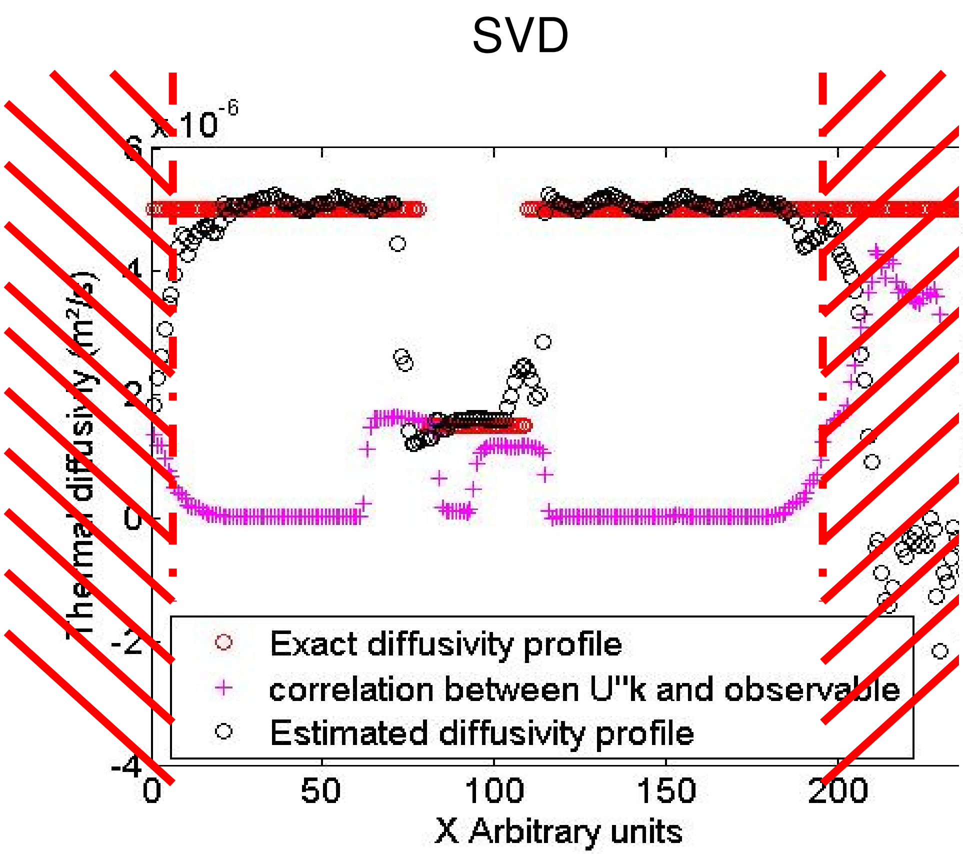
$k = 1..3$

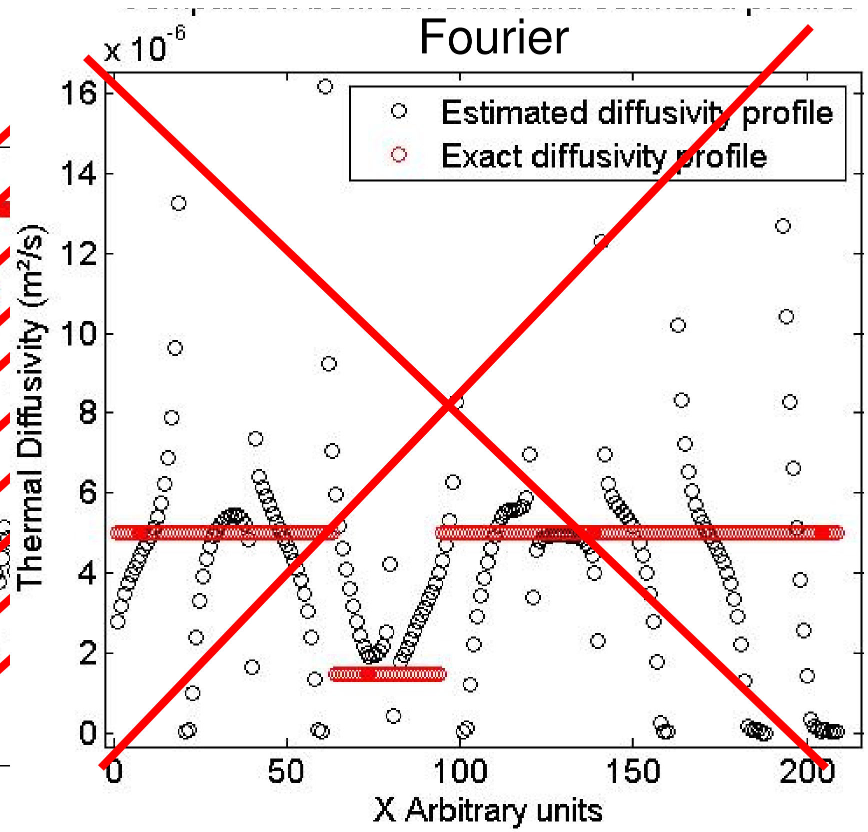
$$a^k(i) = \frac{\tilde{U}_k(i)}{\ddot{U}_k(i)}$$



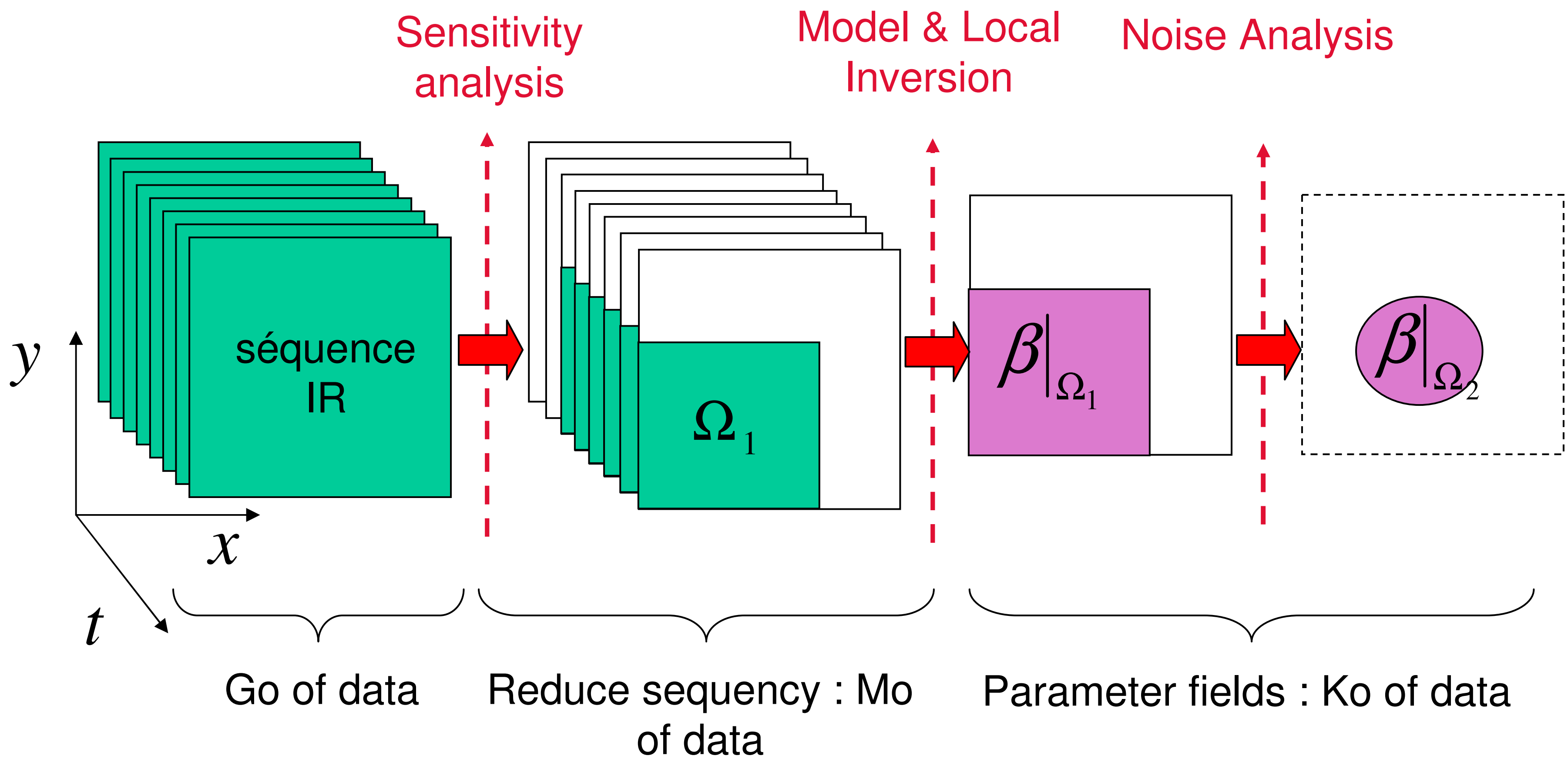
No noise, maximum sensitivity

Simulation results with noise





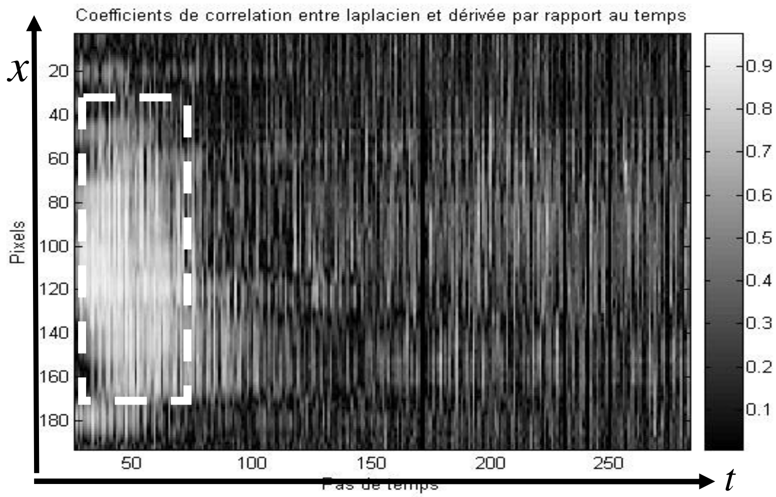
Same kind of correlation approach for the nodal methods



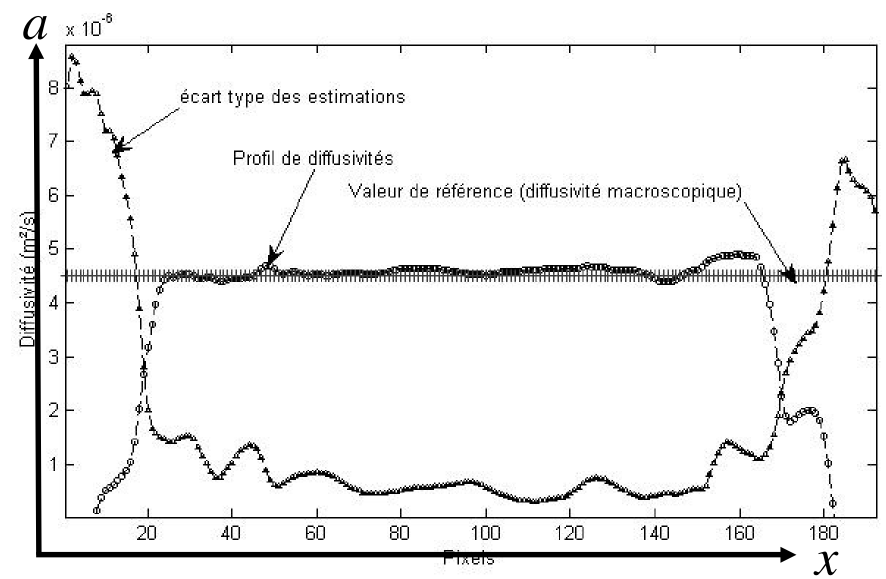
1D Example: ceramic composite medium

$$a_x(x,t)=\left(\int_{u=x-\tau/2}^{x+\tau/2}\frac{\partial^2\tilde{T}(u,t)}{\partial x^2}\cdot\frac{\partial\tilde{T}(u,t)}{\partial t}du\right)/\left(\int_{u=x-\tau/2}^{x+\tau/2}\left(\frac{\partial^2\tilde{T}(u,t)}{\partial x^2}\right)^2du\right)=\rho(x,t)\cdot\frac{\left\|\frac{\partial\tilde{T}(x,t)}{\partial t}\right\|}{\left\|\frac{\partial^2\tilde{T}(x,t)}{\partial x^2}\right\|}$$

$$a_x(x)=\left(\int_t\rho(x,t)\cdot a_x(x,t)dt\right)/\left(\int_t\rho(x,t)dt\right)$$



Correlation factor mapping



Thermal diffusivity profile

Bibliographical Références

J.C. Batsale, J.L. Battaglia, O. Fudym, *Autoregressive algorithms and spatially random flash excitation for 3D non destructive evaluation with infrared cameras*. QIRT Journal **1** 5-20, 2004.

M. Bamford, J.C. Batsale, D. Reungoat, and O. Fudym, T*wo dimensional velocity and diffusion mappings in the case of three dimensional transient diffusion: "Flash" method and infrared images sequences analysis*, 8th Conf. on Quantitative Infrared Thermography, QIRT 2006, Padova, Italy, June 27-30, 2006.

J.L. Barron, D.J. Fleet, S.S. Beauchemin, *Performances of optical flow techniques*, International Journal of Computer Vision, 12, 1994, 43-77. 18

J.V. Beck, K. J. Arnold, *Parameter estimation in engineering and science*, John Wiley and Sons, NY, 1977

I. Philippi, J.C. Batsale, D. Maillet, A. Degiovanni, *Measurement of thermal diffusivity through processing of infrared images* Rev. Sci. Instrum. 66(1) (1995) 182-192

J.R. Shenfelt, R. Luck, R.P. Taylor, J.T. Berry, *Solution to inverse heat conduction problems employing SVD and model reduction*. IJHMT 45 (2002), 67-74.

N. Rajic, *Principal component thermography for flaw contrast enhancement and flaw depth characterisation in composite structures*. Composite structures 58(4) (2002), 521-528.

S. Roux, L. Dos Santos Lucena, *Estimating thermal conductivities from temperature maps,* Inverse Problems in Science and Engineering, Vol 14, No. 8, (2006) 785-800

J.C. Krapez, L. Spagnolo, M. Frieß, H.P. Maier, G. Neuer, *Measurement of in-plane diffusivity in non-homogeneous slabs by applying flash thermography*, International Journal of Thermal Sciences 43 (2004) 967–977

S.V. Patankar, *Numerical heat transfer and fluid flow*, Hemisphere Publishing Corporation, 1980

L. Ljung, *System identification: theory for the user*, Prentice Hall, 1987.

D. Maillet, André S., Batsale J.C., Degiovanni A., Moyne C., *Thermal quadrupoles : Solving the heat equation through integral transforms*, John Wiley, 2000.

X. Maldague, *Theory and practice of infrared technology for Nondestructive Testing*, John Wiley, New-York, 2001.

M. Mühlich, R. Mester, *The role of total least squares in motion analysis*, in: ECCV, Freiburg, Germany (1998) 305-321.

Hadamard, J., *Lectures on Cauchy's Problem in Linear Differential Equations*, Yale University Press, New Haven, CT, 1923.

Ozisik, M.N. and Orlande, H.R.B., *Inverse Heat Transfer: Fundamentals and Applications*, Taylor and Francis, New York, 2000.

Alifanov, O. M., *Inverse Heat Transfer Problems*, Springer-Verlag, New York, 1994.

J.V. Beck and K.J. Arnold, *Parameter estimation in engineering and science*, John Wiley and sons, New York, 1977.

J.V. Beck, B. Blackwell and C.R. St-Clair Jr., *Inverse heat conduction - Ill-posed problems*, John Wiley and sons, New York, 1985.

D. Maillet, I. Philippi, A. Bendada and A. Degiovanni, *Integral Transforms and Parameter Estimation in Diffusive Transfers*, actes de la Second International Conference on *Inverse Problems in Engineering: Theory and Practice* , Le Croisic, France, June 9-14, 1996, editeurs: D. Delaunay, Y. Jarny et K. Woodbury, publié par l'Engineering Foundation, ASME, New York, pp 463-471, 1998.

A. Tikhonov and V. Arsénine, *Méthodes de résolution des problèmes mal-posés*, Editions de Moscou, 1976.

D. Maillet, *Méthodologie de l'estimation de grandeurs en métrologie thermique,* In Métrologie Thermique et Techniques inverses, Vol.1, Presses Universitaires de Perpignan, 1999.

J.C. Batsale, 2005, *Experimental Inversion and Thermal Sensors*, Eurotherm Winter School, Lecture , METTI 2005, Aussois, January 16-21, 2005.

Obrigado…

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