Proceedings of IMECE2007
2007 ASME International Mechanical Engineering Congress & Exposition
November 11-15, 2007, Seattle, Washington, USA

IMECE2007-41412

UNIFIED HYBRID THEORETICAL ANALYSIS OF NONLINEAR CONVECTIVE HEAT TRANSFER

Renato M. Cotta
Laboratory of Transmission and Technology of Heat-LTTC
Mechanical Eng. Dept. - COPPE/UFRJ
Rio de Janeiro, RJ, Brasil
cotta@mecanica.coppe.ufrj.br
Sadik Kakaç
TOBB Economics and Technology University
Ankara, Turkey
sadikakaç@yahoo.com

Carlos Antonio Cabral dos Santos
Laboratory of Solar Energy - LES
Mechanical Eng. Dept. – UFPb
João Pessoa, PB, Brasil
cabral@les.ufpb.br

ABSTRACT

The Generalized Integral Transform Technique (GITT) is extended and unified to handle a wide class of nonlinear convective heat and mass transfer formulations that involve nonlinearities in every each equation and boundary condition coefficients and source terms. The proposed hybrid numerical-analytical approach is applicable to both steady and transient situations, and the integral transformation process is promoted so as to yield an explicit transformed system, which can be more efficiently handled by the well-known numerical algorithms for initial value problems. An application of laminar forced convection in duct flow with temperature dependent thermophysical properties is considered more closely to illustrate the formal solution.

INTRODUCTION

Discrete numerical methods for partial differential formulations in convection-diffusion belong nowadays to the routine work of thermal engineers involved in design and development tasks, and are no longer restricted to research and scientific environments. The various possible approaches are consolidated and readily available in textbooks and reference material, as well as in commercial application packages [1]. Nevertheless, from the research point of view, there is still a strong motivation for the improvement of existing simulation techniques and for the development of novel strategies that benefit from numerical analysis and computer science advances as a whole, such as the advancements on adaptive error control of ordinary and partial differential equations and the dissemination of symbolic computation platforms.

In this context, solution techniques for partial differential equations that exploit the analytical knowledge database and rely on modern symbolic computation platforms have been calling further attention of the research community and offering measurable advantages over the classical numerical approaches in a number of applications. Within this wide research front, we may place the advancement of the Generalized Integral Transform Technique (GITT) for the hybrid numerical-analytical solution of convection-diffusion problems [2-7]. In this case the emphasis is placed on extending the classical integral transform method making it sufficiently flexible to handle problems that are not a priori transformable, such as in the case of problems with nonlinear coefficients in either the equation or the boundary conditions [8-15]. Various classes of nonlinear problems in heat and fluid flow have been handled by the GITT, and among them convective heat transfer problems formulated by either the boundary layer of full Navier-Stokes formulations, for cavity, duct or external flows, here reviewed just for the internal flow situation of closer interest to the application to be later considered [16-30]. Nevertheless, only in a few situations [23-24] the full nonlinear nature of these equations have been dealt with, including not only the usual nonlinear terms that derive from the convective formulation, but also those due to the variable physical properties, especially in their dependence with temperature.

The present work is thus aimed at extending and unifying the integral transform approach in handling convection-diffusion problems with nonlinear behavior in all equation and boundary condition coefficients and source terms. A fairly general formulation is considered that encompasses a wide class of problems that appear in thermal engineering. In light of the possible nonlinear nature of the transient term (or equivalent space variable), an explicit transformed system formulation is preferred, which brings advantages in the numerical computation of the transformed potentials. Also, a flexible filtering solution is allowed for that aims at reducing the relative importance of equation and boundary source terms in the eigenfunction expansions convergence behavior.

The proposed methodology is then illustrated for a fully nonlinear convective heat transfer problem related to laminar tube flow of ordinary liquids with all thermophysical properties expressed with temperature dependence.

PROBLEM FORMULATION & FORMAL SOLUTION

We consider a sufficiently general formulation on convection-diffusion, which encompasses most of the problem statements in convective heat and mass transfer of interest in thermal engineering that have been dealt with via integral transforms (GITT). A set of potentials, $T_k(\mathbf{x},t),\ k=1,2,...M$, dependent on position $\mathbf{x}$ and time t (or equivalent space coordinate) such as temperature, concentrations, velocity components, pressure, etc, are defined in region V with boundary surface S, and obey the following convection-diffusion equations which include nonlinear coefficients in all equation and boundary source terms:

$$w_k^*(\mathbf{x},t,T_l)\frac{\partial T_k(\mathbf{x},t)}{\partial t}+\mathbf{u}(\mathbf{x},t,T_l).\nabla T_k(\mathbf{x},t)=\nabla.k_k^*(\mathbf{x},t,T_l)\nabla T_k(\mathbf{x},t)-$$
$$-d_k^*(\mathbf{x},t,T_l)T_k(\mathbf{x},t)+P_k^*(\mathbf{x},t,T_l),\ \mathbf{x}\in V,\ t>0,\ k,l=1,2,...M \tag{1.a}$$

with initial and boundary conditions

$$T_k(\mathbf{x},0)=f_k(\mathbf{x}),\quad \mathbf{x}\in V \tag{1.b}$$

$$\alpha_k^*(\mathbf{x},t,T_l)T_k(\mathbf{x},t)+\beta_k^*(\mathbf{x},t,T_l)k_k^*(\mathbf{x},t,T_l)\frac{\partial T_k(\mathbf{x},t)}{\partial \mathbf{n}}$$
$$=\phi_k^*(\mathbf{x},t,T_l),\quad \mathbf{x}\in S \tag{1.c}$$

The proposed hybrid numerical-analytical approach, based on integral transforms, starts with the proposition of a formal solution in terms of an eigenfunction expansion of the desired potentials, $T_k(\mathbf{x},t)$, with the corresponding time-dependent expansion coefficients, $A_{k,i}(t)$, to be determined:

$$T_k(\mathbf{x},t)=\sum_{i=1}^{\infty}A_{k,i}(t)\psi_{k,i}(\mathbf{x}) \tag{2}$$

where the eigenfunctions $\psi_{k,i}(\mathbf{x})$, are obtained from a representative eigenvalue problem that contains as much information on the original problem as possible, in the form:

$$\nabla.k_k(\mathbf{x})\nabla\psi_{k,i}(\mathbf{x})+(\mu_{k,i}^2 w_k(\mathbf{x})-d_k(\mathbf{x}))\psi_{k,i}(\mathbf{x})=0,\mathbf{x}\in V \tag{3a}$$

with boundary conditions

$$\alpha_k(\mathbf{x}_l)\psi_{k,i}(\mathbf{x})+\beta_k(\mathbf{x})k_k(\mathbf{x})\frac{\partial\psi_{k,i}(\mathbf{x})}{\partial\mathbf{n}}=0,\quad \mathbf{x}\in S \tag{3b}$$

The coefficients, $w_k(\mathbf{x}), k_k(\mathbf{x}), d_k(\mathbf{x}), \alpha_k(\mathbf{x}), and\ \beta_k(\mathbf{x})$, of the auxiliary problem (3a)-(3b) are expected to include information related to the original nonlinear coefficients in eqs.(1.a)-(1.c). Note that the convection term in eq.(1.a) was not represented in the auxiliary problem (3a)-(3b), since a non-self adjoint eigenvalue problem would result. Although this situation was also considered with some advantages in the GITT literature, for the sake of generality in the present unification attempt, only Sturm-Liouville problems of well-known spectral behavior are considered. Thus, problem (3a)-(3b) offers an orthogonality property to the eigenfunctions which is quite relevant in the present methodology, written as:

$$\int_V w_k(\mathbf{x})\psi_{k,i}(\mathbf{x})\psi_{k,j}(\mathbf{x})dv=\delta_{i,j}N_{k,i} \tag{4a}$$

where the Kronecker delta as usual is equal to 1 for i=j or 0 for i≠j and $N_{k,i}$ are the normalization integrals which are computed from:

$$N_{k,i}=\int_V w_k(\mathbf{x})\psi_{k,i}^2(\mathbf{x})dv \tag{4b}$$

With the aid of eq. (4a), we may operate on the proposed expansion, eq. (2), with the integral operator $\int_V w_k(\mathbf{x})\psi_{k,j}(\mathbf{x})-dv$, to obtain the expansion coefficients:

$$A_{k,j}(t)=\frac{1}{N_{k,j}}\int_V w_k(\mathbf{x})\psi_{k,j}(\mathbf{x})T_k(\mathbf{x},t)dv \tag{5}$$

once all of the terms in the infinite sum of eq.(2) vanish, except that one when i=j. Then, eqs.(2) and (5) offer the integral transformation pair of formulae, called the inverse and the transform:

$$T_k(\mathbf{x},t)=\sum_{i=1}^{\infty}\tilde{\psi}_{k,i}(\mathbf{x})\overline{T}_{k,i}(t),\qquad inverse \tag{6a}$$

$$\overline{T}_{k,i}(t)=\int_V w_k(\mathbf{x})\tilde{\psi}_{k,i}(\mathbf{x})T_k(\mathbf{x},t)dv,\qquad transform \tag{6b}$$

where the normalized eigenfunction is adopted, thus splitting the norm between the two formulae, transform and inverse, in the form:

$$\tilde{\psi}_{k,i}(\mathbf{x}) = \frac{\psi_{k,i}(\mathbf{x})}{\sqrt{N_{k,i}}} \quad (6c)$$

The next step in the GITT approach is then the integral transformation of eqs.(1.a), making use of the transform-inverse pair above defined. The traditional approach would involve the integral transform operation, with the aid of the transformation formulae, eq.(6.b), on both sides of eq.(1.a), which for this fully nonlinear formulation would result in a coupling nonlinear coefficients matrix in the left hand side of the transformed system, due to the nonlinear nature of the transient term coefficient in the original problem, $w_k^*(\mathbf{x},t,T_l)$. From the computational point of view, such implicit nonlinear formulation would require that this matrix be inverted along the numerical integration process for the transformed system, from most of the initial value problem solvers procedure, resulting in increased computational cost. However, before proceeding, it is advantageous to rewrite the problem formulation so as to offer an explicit linear integral transformation of the transient term. Since the final result of the integral transformation shall be the construction of an initial value problem for obtaining the transformed potentials, $\bar{T}_{k,i}(t)$, it is less computationally involved to the associated numerical solution procedure for ordinary differential equations, to deal with an explicit linear system that would not require numerous matrix inversions related to a nonlinear coefficients matrix in the transient term. Therefore, the transient term coefficient can be rewritten as:

$$w_k^*(\mathbf{x},t,T_l) = w_k(\mathbf{x})\frac{w_k^*(\mathbf{x},t,T_l)}{w_k(\mathbf{x})} = w_k(\mathbf{x})C_k^{-1}(\mathbf{x},t,T_l) \quad (7)$$

which results in the new version for eq.(1a):

$$\begin{aligned} & w_k(\mathbf{x})\frac{\partial T_k(\mathbf{x},t)}{\partial t} = \\ & = C_k(\mathbf{x},t,T_l)[\nabla.k_k^*(\mathbf{x},t,T_l)\nabla T_k(\mathbf{x},t) - d_k^*(\mathbf{x},t,T_l)T_k(\mathbf{x},t) \\ & -\mathbf{u}(\mathbf{x},t,T_l).\nabla T_k(\mathbf{x},t) + P_k^*(\mathbf{x},t,T_l)],\ \mathbf{x}\in V,\ t>0 \end{aligned} \quad (8)$$

or simply,

$$w_k(\mathbf{x})\frac{\partial T_k(\mathbf{x},t)}{\partial t} = H_k(\mathbf{x},t,T_l), \quad t>0,\ l,k=1,2,...,M \quad (9a)$$

where,

$$\begin{aligned} & H_k(\mathbf{x},t,T_l) = C_k(\mathbf{x},t,T_l)[\nabla.k_k^*(\mathbf{x},t,T_l)\nabla T_k(\mathbf{x},t) \\ & -d_k^*(\mathbf{x},t,T_l)T_k(\mathbf{x},t) - \mathbf{u}(\mathbf{x},t,T_l).\nabla T_k(\mathbf{x},t) + P_k^*(\mathbf{x},t,T_l)] \end{aligned} \quad (9b)$$

with initial conditions given by eq. (1.b), and also rewritten boundary conditions

$$\alpha_k(\mathbf{x})T_k(\mathbf{x},t) + \beta_k(\mathbf{x})k_k(\mathbf{x})\frac{\partial T_k(\mathbf{x},t)}{\partial \mathbf{n}} = \phi_k(\mathbf{x},t,T_l), \quad \mathbf{x}\in S \quad (9c)$$

where,

$$\begin{aligned} & \phi_k(\mathbf{x},t,T_l) = \phi_k^*(\mathbf{x},t,T_l) + [\alpha_k(\mathbf{x}) - \alpha_k^*(\mathbf{x},t,T_l)]T_k(\mathbf{x},t) \\ & +[\beta_k(\mathbf{x})k_k(\mathbf{x}) - \beta_k^*(\mathbf{x},t,T_l)k_k^*(\mathbf{x},t,T_l)]\frac{\partial T_k(\mathbf{x},t)}{\partial \mathbf{n}} \end{aligned} \quad (9d)$$

The integral transformation process itself is then performed, by operating eq.(9a) on with $\int_V \tilde{\psi}_{k,i}(\mathbf{x}) - dv$, to obtain:

$$\frac{d\bar{T}_{k,i}(t)}{dt} = \int_V \tilde{\psi}_{k,i}(\mathbf{x})H_k(\mathbf{x},t,T_l)dv, \qquad t>0,\ i=1,2,... \quad (10a)$$

In principle, the direct integration of the right hand side of eq.(10a) would give a vector, upon substitution of the inverse formula, eq.(6a), into the nonlinear terms, that would however not contain any information on the boundary source terms, $\phi_k(\mathbf{x},t,T_l)$. Therefore, in order to account for the boundary condition contribution, we first split this rhs in two, as follows:

$$\begin{aligned} & \frac{d\bar{T}_{k,i}(t)}{dt} = \int_V \tilde{\psi}_{k,i}(\mathbf{x})C_k(\mathbf{x},t,T_l)\nabla.k_k^*(\mathbf{x},t,T_l)\nabla T_k(\mathbf{x},t)dv + \\ & \int_V \tilde{\psi}_{k,i}(\mathbf{x})C_k(\mathbf{x},t,T_l)[P_k^*(\mathbf{x},t,T_l) - d_k^*(\mathbf{x},t,T_l)T_k(\mathbf{x},t) \\ & -\mathbf{u}(\mathbf{x},t,T_l).\nabla T_k(\mathbf{x},t)]dv \end{aligned} \quad (10b)$$

The first term in the rhs above can be rewritten following the 2nd Green´s formula, to find:

$$\begin{aligned} & \int_V \tilde{\psi}_{k,i}(\mathbf{x})C_k(\mathbf{x},t,T_l)\nabla.k_k^*(\mathbf{x},t,T_l)\nabla T_k(\mathbf{x},t)dv = \\ & \int_V T_k(\mathbf{x},t)\nabla.k_k^*(\mathbf{x},t,T_l)\nabla[\tilde{\psi}_{k,i}(\mathbf{x})C_k(\mathbf{x},t,T_l)]dv + \\ & \int_S k_k^*(\mathbf{x},t,T_l)\{\tilde{\psi}_{k,i}(\mathbf{x})C_k(\mathbf{x},t,T_l)\frac{\partial T_k(\mathbf{x},t)}{\partial \mathbf{n}} - \\ & T_k(\mathbf{x},t)\frac{\partial[\tilde{\psi}_{k,i}(\mathbf{x})C_k(\mathbf{x},t,T_l)]}{\partial \mathbf{n}}\}dv \end{aligned} \quad (10c)$$

or,

$$\int_V \tilde{\psi}_{k,i}(\mathbf{x})C_k(\mathbf{x},t,T_l)\nabla.k_k^*(\mathbf{x},t,T_l)\nabla T_k(\mathbf{x},t)dv =$$
$$\int_V T_k(\mathbf{x},t)[\nabla.k_k^*\tilde{\psi}_{k,i}\nabla C_k + \nabla.k_k^* C_k\nabla\tilde{\psi}_{k,i}]dv +$$
$$\int_S k_k^*(\mathbf{x},t,T_l)\{C_k(\mathbf{x},t,T_l)[\tilde{\psi}_{k,i}(\mathbf{x})\frac{\partial T_k(\mathbf{x},t)}{\partial\mathbf{n}} - \tag{10d}$$
$$T_k(\mathbf{x},t)\frac{\partial\tilde{\psi}_{k,i}(\mathbf{x})}{\partial\mathbf{n}}] - T_k(\mathbf{x},t)\tilde{\psi}_{k,i}(\mathbf{x})\frac{\partial C_k(\mathbf{x},t,T_l)}{\partial\mathbf{n}}\}dv$$

Therefore, the transformed system can be written in the short form below:

$$\frac{d\overline{T}_{k,i}(t)}{dt} = \overline{h}_{k,i}(t,\overline{T}_{l,j}), \qquad t>0,\ k=1,2,...,M,\ i=1,2,... \tag{11a}$$

where the vector $\overline{h}_{k,i}(t,\overline{T}_{l,j})$, is formed by the three contributions below:

$$\overline{h}_{k,i}(t,\overline{T}_{l,j}) = \overline{h}^*_{k,i}(t,\overline{T}_{l,j}) + \overline{q}_{k,i}(t,\overline{T}_{l,j}) + \overline{g}_{k,i}(t,\overline{T}_{l,j}) \tag{11b}$$

with,

$$\overline{h}^*_{k,i}(t,\overline{T}_{l,j}) = \int_V \tilde{\psi}_{k,i}(\mathbf{x})C_k(\mathbf{x},t,T_l)[P_k^*(\mathbf{x},t,T_l)$$
$$-d_k^*(\mathbf{x},t,T_l)T_k(\mathbf{x},t) - \mathbf{u}(\mathbf{x},t,T_l).\nabla T_k(\mathbf{x},t)]dv \tag{11c}$$

$$\overline{q}_{k,i}(t,\overline{T}_{l,j}) = \int_V T_k(\mathbf{x},t)[\nabla.k_k^*\tilde{\psi}_{k,i}\nabla C_k + \nabla.k_k^* C_k\nabla\tilde{\psi}_{k,i}]dv \tag{11d}$$

$$\overline{g}_{k,i}(t,\overline{T}_{l,j}) = \int_S k_k^*(\mathbf{x},t,T_l)\{C_k(\mathbf{x},t,T_l)[\tilde{\psi}_{k,i}(\mathbf{x})\frac{\partial T_k(\mathbf{x},t)}{\partial\mathbf{n}} -$$
$$T_k(\mathbf{x},t)\frac{\partial\tilde{\psi}_{k,i}(\mathbf{x})}{\partial\mathbf{n}}] - T_k(\mathbf{x},t)\tilde{\psi}_{k,i}(\mathbf{x})\frac{\partial C_k(\mathbf{x},t,T_l)}{\partial\mathbf{n}}\}ds \tag{11e}$$

The coefficient vector containing the eigenfunction divergent contribution may also be rewritten more conveniently for computational purpose in the form:

$$\overline{q}_{k,i}(t,\overline{T}_{l,j}) = \int_V T_k(\mathbf{x},t)[\nabla.k_k^*\nabla C_k + \nabla k_k^*.\nabla C_k]\tilde{\psi}_{k,i}dv +$$
$$\int_V T_k(\mathbf{x},t)(2\gamma_k\nabla C_k + C_k\nabla\gamma_k).(k_k\nabla\tilde{\psi}_{k,i})dv + \tag{11f}$$
$$\int_V T_k(\mathbf{x},t)\gamma_k C_k\nabla.(k_k\nabla\tilde{\psi}_{k,i})dv$$

The eigenvalue problem, eq.(3a), can be employed to further simplify this vector, as:

$$\overline{q}_{k,i}(t,\overline{T}_{l,j}) = \int_V T_k(\mathbf{x},t)[\nabla.k_k^*\nabla C_k + \nabla k_k^*.\nabla C_k + \gamma_k C_k d_k]\tilde{\psi}_{k,i}(\mathbf{x})dv +$$
$$\int_V T_k(\mathbf{x},t)(2\gamma_k\nabla C_k + C_k\nabla\gamma_k).(k_k\nabla\tilde{\psi}_{k,i}(\mathbf{x}))dv -$$
$$\mu_i^2\int_V T_k(\mathbf{x},t)\gamma_k C_k w_k\tilde{\psi}_{k,i}(\mathbf{x})dv \tag{11g}$$

where,

$$\gamma_k(\mathbf{x},t,T_l) = \frac{k_k^*(\mathbf{x},t,T_l)}{k_k(\mathbf{x})} \tag{11h}$$

The boundary source term contribution can be written explicitly by manipulating the two boundary conditions, for the original and auxiliary problems, eqs.(9d) and (3b), respectively, to yield:

$$\overline{g}_{k,i}(t,\overline{T}_{l,j}) = \int_S \gamma_k C_k\phi_k(\mathbf{x},t,T_l)[\frac{\tilde{\psi}_{k,i}(\mathbf{x}) - k_k(\mathbf{x})\frac{\partial\tilde{\psi}_{k,i}(\mathbf{x})}{\partial\mathbf{n}}}{\alpha_k(\mathbf{x})+\beta_k(\mathbf{x})}]ds$$
$$-\int_S k_k^*(\mathbf{x},t,T_l)T_k(\mathbf{x},t)\tilde{\psi}_{k,i}(\mathbf{x})\frac{\partial C_k(\mathbf{x},t,T_l)}{\partial\mathbf{n}}ds \tag{11i}$$

Though formal and exact, the above manipulation to account for the boundary conditions source terms introduces some additional complexity due to the requirement of evaluating derivatives of the nonlinear coefficient $C_k(\mathbf{x},t,T_l)$. Alternatively, one may prefer a more straightforward approach to find the contribution of the boundary source terms, as now described. Starting from eq.(10.b), we sum and subtract the contribution of the actual flux divergent term, i.e.:

$$\frac{d\overline{T}_{k,i}(t)}{dt} = \int_V \tilde{\psi}_{k,i}(\mathbf{x})[C_k(\mathbf{x},t,T_l)-1]\nabla.k_k^*(\mathbf{x},t,T_l)\nabla T_k(\mathbf{x},t)dv +$$
$$\int_V \tilde{\psi}_{k,i}(\mathbf{x})\nabla.k_k^*(\mathbf{x},t,T_l)\nabla T_k(\mathbf{x},t)dv +$$
$$\int_V \tilde{\psi}_{k,i}(\mathbf{x})C_k(\mathbf{x},t,T_l)[P_k^*(\mathbf{x},t,T_l) - d_k^*(\mathbf{x},t,T_l)T_k(\mathbf{x},t)$$
$$-\mathbf{u}(\mathbf{x},t,T_l).\nabla T_k(\mathbf{x},t)]dv \tag{11j}$$

Now, the second Green formula is applied solely to the second term in the right hand side, which corresponds to the integral transformation of the original flux divergence term, in the form:

$$\frac{d\overline{T}_{k,i}(t)}{dt}=\int_V \tilde{\psi}_{k,i}(\mathbf{x})[C_k(\mathbf{x},t,T_l)-1]\nabla.k_k^*(\mathbf{x},t,T_l)\nabla T_k(\mathbf{x},t)dv+$$
$$\int_V T_k(\mathbf{x},t)\nabla.k_k^*(\mathbf{x},t,T_l)\nabla\tilde{\psi}_{k,i}(\mathbf{x})dv+$$
$$\int_S k_k^*(\mathbf{x},t,T_l)[\tilde{\psi}_{k,i}(\mathbf{x})\frac{\partial T_k(\mathbf{x},t)}{\partial\mathbf{n}}-T_k(\mathbf{x},t)\frac{\partial\tilde{\psi}_{k,i}(\mathbf{x})}{\partial\mathbf{n}}]ds+$$
$$\int_V \tilde{\psi}_{k,i}(\mathbf{x})C_k(\mathbf{x},t,T_l)[P_k^*(\mathbf{x},t,T_l)-d_k^*(\mathbf{x},t,T_l)T_k(\mathbf{x},t)$$
$$-\mathbf{u}(\mathbf{x},t,T_l).\nabla T_k(\mathbf{x},t)]dv \tag{11k}$$

Therefore, the alternative transformed system can be written in the short form below:

$$\frac{d\overline{T}_{k,i}(t)}{dt}=\hat{\hat{h}}_{k,i}(t,\overline{T}_{l,j}),\qquad t>0,\ k=1,2,...,M,\ i=1,2,... \tag{11l}$$

where the vector $\hat{\hat{h}}_{k,i}(t,\overline{T}_{l,j})$, is now formed by the three new contributions below:

$$\hat{\hat{h}}_{k,i}(t,\overline{T}_{l,j})=\hat{\hat{h}}^*_{k,i}(t,\overline{T}_{l,j})+\hat{\hat{q}}_{k,i}(t,\overline{T}_{l,j})+\hat{\hat{g}}_{k,i}(t,\overline{T}_{l,j}) \tag{11m}$$

with,

$$\hat{\hat{h}}^*_{k,i}(t,\overline{T}_{l,j})=\int_V \tilde{\psi}_{k,i}(\mathbf{x})[C_k(\mathbf{x},t,T_l)-1]\nabla.k_k^*(\mathbf{x},t,T_l)\nabla T_k(\mathbf{x},t)dv+$$
$$\int_V \tilde{\psi}_{k,i}(\mathbf{x})C_k(\mathbf{x},t,T_l)[P_k^*(\mathbf{x},t,T_l)-d_k^*(\mathbf{x},t,T_l)T_k(\mathbf{x},t)$$
$$-\mathbf{u}(\mathbf{x},t,T_l).\nabla T_k(\mathbf{x},t)]dv \tag{11n}$$

$$\hat{\hat{q}}_{k,i}(t,\overline{T}_{l,j})=\int_V T_k(\mathbf{x},t)\nabla.k_k^*(\mathbf{x},t,T_l)\nabla\tilde{\psi}_{k,i}(\mathbf{x})dv \tag{11o}$$

$$\hat{\hat{g}}_{k,i}(t,\overline{T}_{l,j})=\int_S k_k^*(\mathbf{x},t,T_l)[\tilde{\psi}_{k,i}(\mathbf{x})\frac{\partial T_k(\mathbf{x},t)}{\partial\mathbf{n}}-T_k(\mathbf{x},t)\frac{\partial\tilde{\psi}_{k,i}(\mathbf{x})}{\partial\mathbf{n}}]ds \tag{11p}$$

The alternative transformed system offered by eqs.(11l)-(11p) involves more stratightforward final expressions, especially in avoiding derivatives of the nonlinear coefficients in the original transient term. Eqs.(11a) or (11l) require the transformed initial conditions for each potential, upon integral transformation of eq.(1.b) with $\int_V w_k(\mathbf{x})\tilde{\psi}_{k,i}(\mathbf{x})-dv$, to find:

$$\overline{T}_{k,i}(0)=\int_V w_k(\mathbf{x})\tilde{\psi}_{k,i}(\mathbf{x})\,f_k(\mathbf{x})dv \tag{11q}$$

Eqs.(11l) to (11q) form an infinite coupled system of nonlinear ordinary differential equations for the transformed potentials, $\overline{T}_{k,i}(t)$, which is unlikely to be analytically solvable. Nevertheless, fairly reliable algorithms are readily available to numerically handle this ODE system, after truncation to a sufficiently large finite order. The *Mathematica* system [31] provides the routine NDSolve for solving stiff ODE systems such as the one here considered, under automatic relative error control. Once the transformed potentials have been numerically computed, the *Mathematica* routine automatically provides an interpolating function object that approximates the t variable behavior of the solution in a continuous form. Then, the inversion formula can be recalled to yield the potential field representation at any desired position $\mathbf{x}$ and time t (or equivalent space coordinate).

FILTERING SOLUTION

The formal solution above derived provides the basic working expressions for the integral transform method. However, for an improved computational performance, it is always recommended to reduce the importance of the equation and boundary source terms so as to enhance the eigenfunction expansions convergence behavior.

One possible approach for achieving this goal is the proposition of analytical filtering solutions, which essentially remove information from the source terms into a desirably simple analytical expression. Several different alternative filters may be proposed for the same problem, and the user experience may be quite helpful in finding the right combination of analytical involvement and numerical improvement. Nevertheless, the filter is in general means proposed as:

$$T_k(\mathbf{x},t)=T_k^*(\mathbf{x},t)+T_{k,f}(\mathbf{x};t) \tag{12}$$

where the variable t is a parameter in the filter solution proposition, $T_{k,f}(\mathbf{x};t)$.

The net effect of the filter is to provide a new filtered problem, with reduced importance of the original problem source terms, written as:

$$w_k(\mathbf{x})\frac{\partial T_k^*(\mathbf{x},t)}{\partial t}=$$
$$=C_k(\mathbf{x},t,T_l)[\nabla.k_k^*(\mathbf{x},t,T_l)\nabla T_k^*(\mathbf{x},t)-d_k^*(\mathbf{x},t,T_l)T_k^*(\mathbf{x},t) \tag{13a}$$
$$-\mathbf{u}(\mathbf{x},t,T_l).\nabla T_k^*(\mathbf{x},t)+P_{k,f}(\mathbf{x},t,T_l)],\ \mathbf{x}\in V,\ t>0$$

where the filtered source term is given by

$$P_{k,f}(\mathbf{x},t,T_l) = P_k^*(\mathbf{x},t,T_l) + \nabla . k_k^*(\mathbf{x},t,T_l)\nabla T_{k,F}(\mathbf{x},t)$$
$$-d_k^*(\mathbf{x},t,T_l)T_{k,f}(\mathbf{x},t) - \mathbf{u}(\mathbf{x},t,T_l).\nabla T_{k,f}(\mathbf{x},t) - \frac{w_k(\mathbf{x})}{C_k}\frac{\partial T_{k,f}(\mathbf{x},t)}{\partial t} \tag{13b}$$

with initial and boundary conditions

$$T_k^*(\mathbf{x},0) = f_k^*(\mathbf{x}) = f_k(\mathbf{x}) - T_{k,f}(\mathbf{x},0), \quad \mathbf{x} \in V \tag{13c}$$

$$\alpha_k(\mathbf{x})T_k^*(\mathbf{x},t) + \beta_k(\mathbf{x})k_k(\mathbf{x})\frac{\partial T_k^*(\mathbf{x},t)}{\partial \mathbf{n}} = \phi_{k,f}(\mathbf{x},t,T_l), \; \mathbf{x} \in S \tag{13d}$$

where,

$$\phi_{k,f}(\mathbf{x},t,T_l) = \phi_k(\mathbf{x},t,T_l) - \alpha_k(\mathbf{x})T_{k,f}(\mathbf{x},t) - \beta_k(\mathbf{x})k_k(\mathbf{x})\frac{\partial T_{k,f}(\mathbf{x},t)}{\partial \mathbf{n}} \tag{13e}$$

Thus, the above formal solution applies directly to the following filtered problem, once the initial conditions, the equation and boundary source terms have been adequately substituted.

APPLICATION

We consider forced convection heat transfer inside a circular tube for incompressible laminar flow of a Newtonian liquid with temperature dependent thermophysical properties, including viscosity, thermal capacitance, and thermal conductivity. The tube is subjected to a prescribed uniform wall heat flux, with uniform inlet temperature and negligible viscous dissipation effects. This problem has a strong practical motivation [32] with a renewed interest due to more recent applications in forced convection such as microchannels and nanofluids [33].

The related energy equation and inlet and boundary conditions are written as:

$$\rho(T)c_p(T)u(r,T)\frac{\partial T(r,z)}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left[rk(T)\frac{\partial T(r,z)}{\partial r}\right], \; 0 < r < r_w, \; z > 0 \tag{14a}$$

$$T(r,0) = T_0, \quad 0 \le r \le r_w \tag{14b}$$

$$\frac{\partial T(r,z)}{\partial r} = 0, \;\; r = 0; \quad -k(T)\frac{\partial T(r,z)}{\partial r} = -q_w, \;\; r = r_w, z > 0 \tag{14c}$$

where the temperature dependent fully developed velocity profile is obtained by direct integration of the momentum equation [34]:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\mu(T)\frac{\partial u(r,z)}{\partial r}\right] = \frac{dp(z)}{dz}, \quad 0 < r < r_w, \; z > 0 \tag{15a}$$

$$\frac{\partial u(r,z)}{\partial r} = 0, \quad r = 0; \;\; u(r,z) = 0, \;\; r = r_w, \quad z > 0 \tag{15b}$$

The following dimensionless groups are then defined:

$$R = \frac{r}{r_w}, \quad Z = \frac{\alpha_0 \, z}{u_0 \, r_w^2}, \quad U(R,Z) = \frac{u(r,z)}{u_0},$$
$$U_{fd}(R) = \frac{u_{fd}(r)}{u_0} = 2(1-R^2), \;\; \gamma(\theta) = \frac{k(T)}{k_0}, \;\; \alpha_0 = \frac{k_0}{\rho_0 c_{p,0}}, \tag{16}$$
$$C(\theta) = \frac{\rho_0 c_{p,0} u_{fd}(r)}{\rho(T)c_p(T)u(r,T)}, \quad \theta(R,Z) = \frac{T(r,z) - T_0}{q_w r_w / k_0}$$

and the problem formulation in dimensionless form is given as

$$RU_{fd}(R)\frac{\partial \theta(R,Z)}{\partial Z} = C(\theta)\frac{\partial}{\partial R}\left[R\gamma(\theta)\frac{\partial \theta(R,Z)}{\partial R}\right], \quad 0 < R < 1, \; Z > 0 \tag{17a}$$

$$\theta(R,0) = 0, \quad 0 \le R \le 1 \tag{17b}$$

$$\frac{\partial \theta(R,Z)}{\partial R} = 0, \; R = 0; \quad \gamma(\theta)\frac{\partial \theta(R,Z)}{\partial R} = 1, \;\; R = 1, \quad Z > 0 \tag{17c}$$

The following filtering solution is proposed:

$$\theta_f(R) = \frac{R^2}{2} \tag{18a}$$

with

$$\theta(R,Z) = \theta^*(R,Z) + \theta_f(R) \tag{18b}$$

Then the problem formulation becomes:

$$RU_{fd}(R)\frac{\partial \theta^*(R,Z)}{\partial Z} = C(\theta)\frac{\partial}{\partial R}\left[R\gamma(\theta)\frac{\partial \theta^*}{\partial R}\right] + P_f(\theta^*), \; 0 < R < 1, \; Z > 0 \tag{19a}$$

$$\theta^*(R,0) = -\frac{R^2}{2}, \quad 0 \le R \le 1 \tag{19b}$$

$$\frac{\partial \theta^*(R,Z)}{\partial R} = 0, \; R = 0; \quad \frac{\partial \theta^*(R,Z)}{\partial R} = \left(\frac{1}{\gamma(\theta)} - 1\right), \;\; R = 1, \quad Z > 0 \tag{19c}$$

where,

$$P_f(\theta^*) = C(\theta)\left[2R\gamma(\theta) + R^2\frac{\partial\gamma}{\partial\theta}\left(\frac{\partial\theta^*}{\partial R} + R\right)\right] \tag{19d}$$

The auxiliary problem is chosen as:

$$\frac{d}{dR}\left[R\frac{d\psi_i(R)}{dR}\right] + \mu_i^2 R\psi_i(R) = 0, \quad 0 < R < 1 \tag{20a}$$

$$\frac{d\psi_i(R)}{dR} = 0, \quad R = 0; \quad \frac{d\psi_i(R)}{dR} = 0, \quad R = 1 \tag{20b}$$

which yields the solution

$$\psi_i(R) = J_0(\mu_i R) \tag{20c}$$

with eigenvalues obtained from

$$J_1(\mu_i) = 0, \quad i = 0,1,2,... \tag{20d}$$

where $\mu_0 = 0$ is also an eigenvalue, and the normalization integral is given by

$$N_i = \frac{1}{2}J_0^2(\mu_i) \tag{20e}$$

while the normalized eigenfunctions result in

$$\tilde{\psi}_i(R) = \sqrt{2}\frac{J_0(\mu_i R)}{J_0(\mu_i)} \tag{20f}$$

The integral transform pair is then given by:

$$\theta^*(R,Z) = \sum_{i=0}^{\infty}\tilde{\psi}_i(R)\bar{\theta}_i(Z), \qquad \textit{inverse} \tag{21a}$$

$$\bar{\theta}_i(Z) = \int_0^1 R\tilde{\psi}_i(R)\theta^*(R,Z)dR, \qquad \textit{transform} \tag{21b}$$

The integral transformation process leads to the following ODE system:

$$\sum_{j=1}^{\infty} a_{i,j}\frac{d\bar{\theta}_j(Z)}{dZ} = \hat{\bar{h}}_i(Z,\bar{\theta}_l), \quad Z > 0,\ i,j,l = 0,1,2,... \tag{22a}$$

$$\bar{\theta}_i(0) = \bar{f}_i \tag{22b}$$

where,

$$\hat{\bar{h}}_i^*(Z,\bar{\theta}_j) = \int_0^1 [C(\theta)-1]\frac{\partial}{\partial R}\left(R\gamma(\theta)(\frac{\partial\theta^*}{\partial R} + R)\right)\tilde{\psi}_i(R)dR \tag{22c}$$

$$\hat{\bar{q}}_i(Z,\bar{\theta}_j) = \int_0^1 [\frac{\partial}{\partial R}\left(R\gamma(\theta)\frac{\partial\tilde{\psi}_i(R)}{\partial R}\right)][\theta^*(R,Z) + \frac{R^2}{2}]dR \tag{22d}$$

$$\hat{\bar{g}}_i(Z,\bar{\theta}_j) = \tilde{\psi}_i(1) \tag{22e}$$

where,

$$\frac{\partial\gamma(\theta)}{\partial R} = \frac{\partial\gamma(\theta)}{\partial\theta}\frac{\partial\theta}{\partial R} \tag{22f}$$

or in terms of the more detailed expressions

$$\hat{\bar{q}}_i(Z,\bar{\theta}_j) = \int_0^1 [R(\theta^*(R,Z) + \frac{R^2}{2})\frac{\partial\gamma(\theta)}{\partial\theta}](\frac{\partial\theta^*(R,Z)}{\partial R} + R)\frac{d\tilde{\psi}_i(R)}{dR}dR$$
$$-\mu_i^2\int_0^1 R\gamma(\theta)[\theta^*(R,Z) + \frac{R^2}{2}]\tilde{\psi}_i(R)dR + \tag{22g}$$

with the transformed inlet conditions

$$\bar{f}_i = -\frac{1}{2}\int_0^1 R^3\tilde{\psi}_i(R)dR \tag{22h}$$

Also, the linear coefficients matrix in the transient term is readily computed in analytic form, and inverted only once to provide the desired explicit transformed system for numerical solution. The coefficients are obtained from:

$$a_{i,j} = \int_0^1 RU_{fd}(R)\tilde{\psi}_i(R)\tilde{\psi}_j(R)dR \tag{22i}$$

The dimensionless velocity field is given by direct integration of the momentum equation in the form:

$$U(R,Z) = \frac{1}{2}\frac{\int_0^1 \frac{R'}{\Lambda(\theta)}dR'}{\int_0^1 R'\int_{R'}^1 \frac{R''}{\Lambda(\theta)}dR''dR'} \tag{23a}$$

where the dimensionless viscosity is written as,

$$\Lambda(\theta) = \frac{\mu(T)}{\mu_0} \tag{23b}$$

RESULTS AND DISCUSSION

The proposed approach was implemented in the mixed symbolic-numerical platform *Mathematica* 5.2 [31] and a few representative results were obtained to illustrate the convergence behavior of the eigenfunction expansions. An actual physical situation dealing with internal forced convection of water-alumina nanofluids was considered [35], and the code was employed to investigate the influence of variable thermophysical properties in the heat transfer enhancement attributed to nanofluids undergoing forced convective heating. Here we selected one case employed in the validation with pure water laminar flow with the following pertinent data:

$$r_w = 0.00315\text{ m};\quad q_w = 6891.3\text{ W/m}^2;\quad L = 2.45\text{ m};$$
$$u_0 = 0.159\text{ m/s};\quad T_0 = 21.9\,^{\circ}\text{C};\quad k_0 = 0.6\text{ W / m}\,^{\circ}\text{C};$$
$$\alpha_0 = 1.436\,\text{x}10^{-7}\,\text{m}^2\text{ / s};\quad \nu_0 = 9.584\,\text{x}10^{-7}\,\text{m}^2\text{ / s}$$

The resulting Reynolds number is around Re=1380 and the Prandtl number is Pr=6.67. All the thermophysical properties were allowed to vary with temperature, including viscosity and its corresponding effect on the velocity field. A few selected positions at the external wall along the tube were taken corresponding to thermocouple locations, and are here used to illustrate the convergence behavior of the eigenfunction expansion implemented.

Thus, Table 1 shows the convergence behavior of the dimensionless duct wall temperature at the chosen dimensionless axial positions. The maximum system truncation order is taken as N=10 and NI=38 segments are employed in the semi-analytical integration of the system coefficients vectors [36]. Also shown in the last column are the results from an approximate numerical solution obtained by linearizing the velocity field with the temperature distribution obtained from the linear problem formulation while retaining the other thermophysical properties as temperature dependent. This approximate formulation was solved with the aid of the *Mathematica* system routine NDSolve, which implements a numerical solution via the Method of Lines [31].

Table 1- Convergence of dimensionless duct wall temperature at different axial positions, Z (N<10, NI=38 segments).

Z N	2	4	6	8	10	Num.*
0.0013	0.1366	0.0968	0.0936	0.0951	0.0958	0.0838
0.0179	0.2629	0.2749	0.2782	0.2783	0.2782	0.2823
0.0353	0.3453	0.3633	0.3642	0.3640	0.3639	0.3662
0.0699	0.4686	0.4837	0.4836	0.4832	0.4830	0.4848
0.1080	0.5762	0.5866	0.5860	0.5855	0.5852	0.5874
0.1480	0.6733	0.6800	0.6792	0.6786	0.6782	0.6812
0.1807	0.7452	0.7499	0.7490	0.7483	0.7479	0.7516
0.2180	0.8229	0.8261	0.8250	0.8242	0.8237	0.8286

(*) NDSolve routine – Method of Lines (linearized velocity field) [31]

Clearly, the integral transform results with truncation orders up to 10, already offer a convergence to the third decimal digit in the dimensionless wall temperature along the duct length. On the other hand, the simplified formulation and numerical solution does not seem to offer accurate results at regions close to the inlet, though improving to a two digits agreement with the GITT solution for regions closer to the channel outlet.

Figure 1 below illustrates the dimensionless temperature radial distributions along the channel length, for the same axial locations as considered in Table 1, which are here represented by colors ranging from pure blue to pure red (Z=0.0013, 0.0179, 0.0353, 0.0699, 0.1080, 0.1480, 0.1807, 0.2180). The solid lines correspond to the full nonlinear formulation here considered while the dashed lines are obtained from the classical linear formulation of Graetz problem. As expected the deviations are more significant within the regions of larger temperature gradients, corresponding to regions closer to the wall and as the fluid heating progresses. Also, the heat transfer enhancement effect may be observed in the reduction of the duct wall temperatures as the nonlinear properties are accounted for, especially due to the reduction of the viscosities close to the hotter duct wall, with the subsequent fluid acceleration in this region.

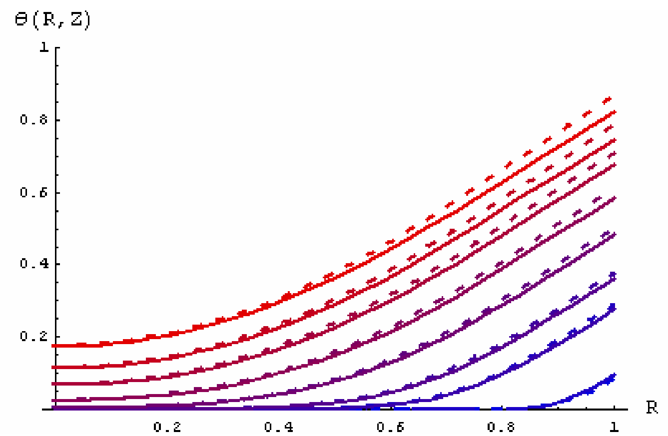

Figure 1- Dimensionless radial temperature distributions for linear (dashed lines) and nonlinear (solid lines) formulations and axial positions increasing from blue to red (Z=0.0013, 0.0179, 0.0353, 0.0699, 0.1080, 0.1480, 0.1807, 0.2180).

CLOSING REMARKS

The present analysis unifies previous developments on the hybrid numerical-analytical approach named the Generalized Integral Transform Technique (GITT), as applied to the solution of convective heat and mass transfer problems in various different situations of cavity, external and duct flows.

The proposed formulation includes nonlinearities in all equation and boundary conditions coefficients and source terms, so as to allow for the direct simplification of the final derived expressions for every each specific application. Also, the methodology here advanced introduces a strategy to avoid nonlinear implicit transformed ODE systems, by rewriting the problem formulation in explicit form with respect to the highest derivative of the dependent variable not eliminated by integral transformation. In this way, an explicit transformed system results which allows for computational savings in the available numerical solvers for initial value problems, since the inversion of a nonlinear coefficients matrix is avoided.

Finally, an example of thermally developing forced convection in laminar tube flow of liquids is considered for illustrating the use of the derived expressions, involving all of the thermophysical properties, viscosity, thermal conductivity and thermal capacitance, with temperature dependence.

The hybrid solutions here developed are part of a wider on-going effort in the construction of an unified integral transforms simulation platform (UNIT Project) for diffusion and convection-diffusion problems [37].

ACKNOWLEDGEMENTS

The authors would like to acknowledge the partial financial support provided by the Brazilian agencies and programs, FAPERJ, PRONEX (Núcleo de Excelência em Turbulência), CNPq and CAPES.

NOMENCLATURE

c_p- specific heat of the fluid [J/kgC]; k -thermal conductivity of the fluid [$W.\ m^{-1}\ .K^{-1}$], Pr- Prandtl number; q_w - imposed heat flux at duct wall[W/m^2]; r_w – tube radius [m]; Re- Reynolds number, T_0- inlet temperature [°C]; T- fluid temperature [°C]; u- fully developed velocity [$m.s^{-1}$]; u_0- average velocity [$m.s^{-1}$]; U-dimensionless longitudinal velocity; z -longitudinal coordinate [m]; Z- dimensionless longitudinal coordinate; r- radial coordinate [m]; R- dimensionless radial coordinate; **Greek letters:** •- thermal diffusivity for the fluid [$m^2.s^{-1}$]; •- dimensionless temperature (fluid); •- kinematic viscosity [$m^2.s^{-1}$]; μ- absolute viscosity [kg.$m.s^{-1}$]; ρ- fluid specific mass [kg/m^3]; Λ – dimensionless viscosity;

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