Hybrid Integral Transforms in Convection-Diffusion

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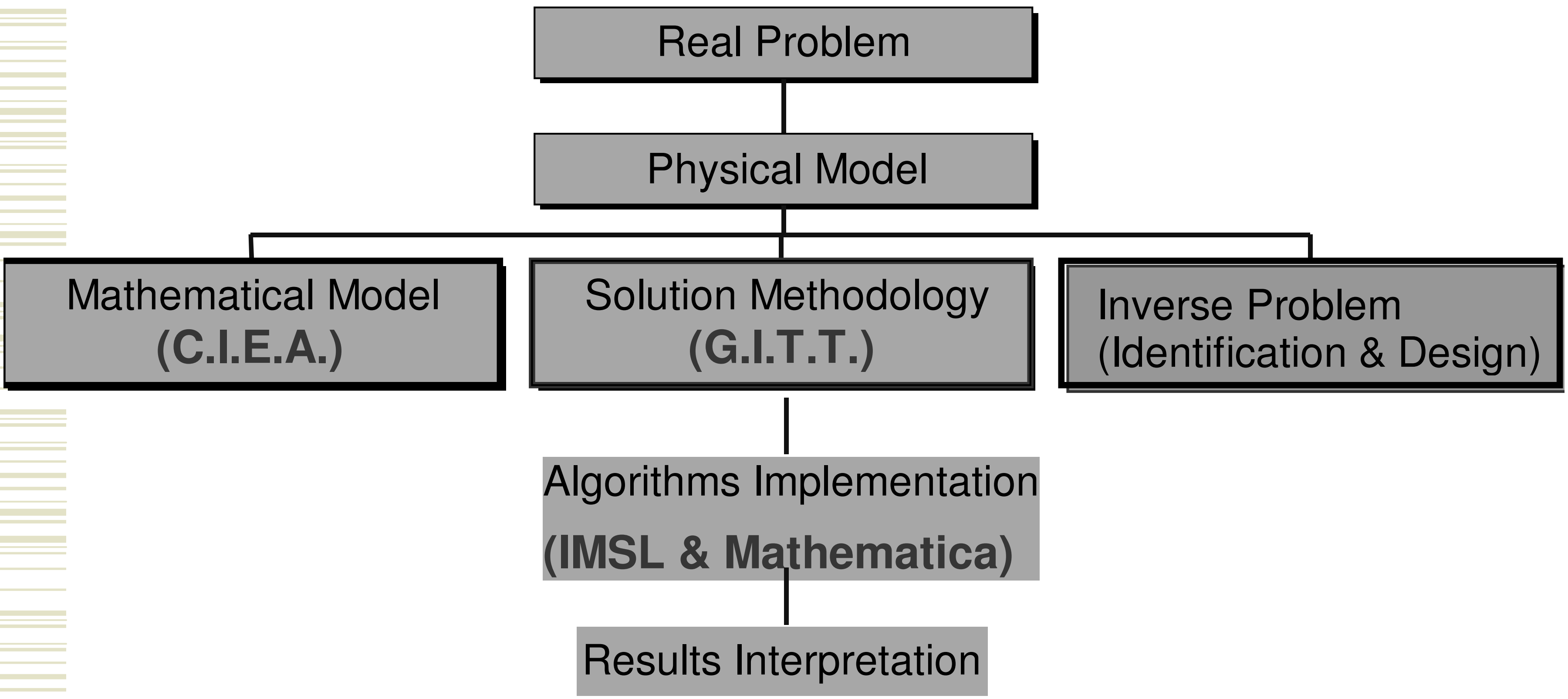
PROPFIS
Escola Sul-Americana em Identificação de Propriedades Físicas em Transferência de Calor e Massa
Rio de Janeiro, Junho de 2005

LTTC
LABORATÓRIO DE TRANSMISSÃO E TECNOLOGIA DO CALOR

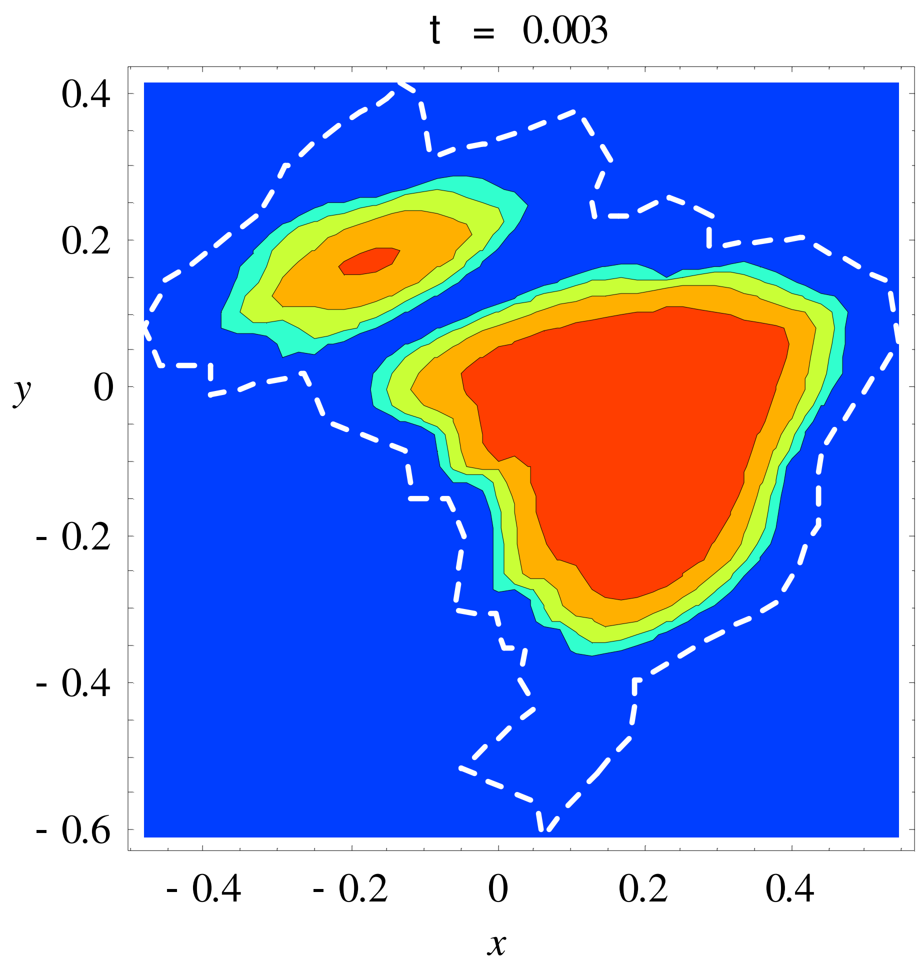
The Hybrid Approach

The Simulation Process



Motivation

- Develop improved lumped-differential formulations in heat and fluid flow.
- Advance a hybrid numerical-analytical solution methodology for PDE's.
- Exploit new concepts on algorithm implementation, based on mixed symbolic-numerical computation.
- Construct new algorithms for inverse problem analysis based on such hybrid paths.



Hybrid Tools

- The Coupled Integral Equations Approach (Improved Formulations).
- The Generalized Integral Transform Technique (Hybrid Methods).
- The *Mathematica* System (Mixed Computations)
- Inverse Problems (Identification & Design)

Hybrid Numerical-Analytical Methods

The Generalized Integral Transform Technique

GITT

Goals

- Reduce computational costs in the solution of multidimensional PDE's, with respect to classical discrete approaches.
- Offer automatic global accuracy control, allowing for the establishment of a benchmarks database in heat and fluid flow.
- Fully exploit the analytic nature of a hybrid approach for implementation in mixed symbolic-numerical computation platforms.

The Generalized Integral Transform Technique - GITT

- Choose the associated eigenvalue problem.
- Develop the integral transform pair.
- Integral transform the original PDE.
- Numerically (or analytically) solve the resulting coupled ODE system for the transformed potentials.
- Recall the analytical inversion formula to reconstruct the hybrid solution of the desired potential.

Classes of Problems
(Linear and Nonlinear)

- Diffusion
- Convection-Diffusion
- Eigenvalue Problems
- Boundary Layer Equations
- Navier-Stokes Equations

Advantages

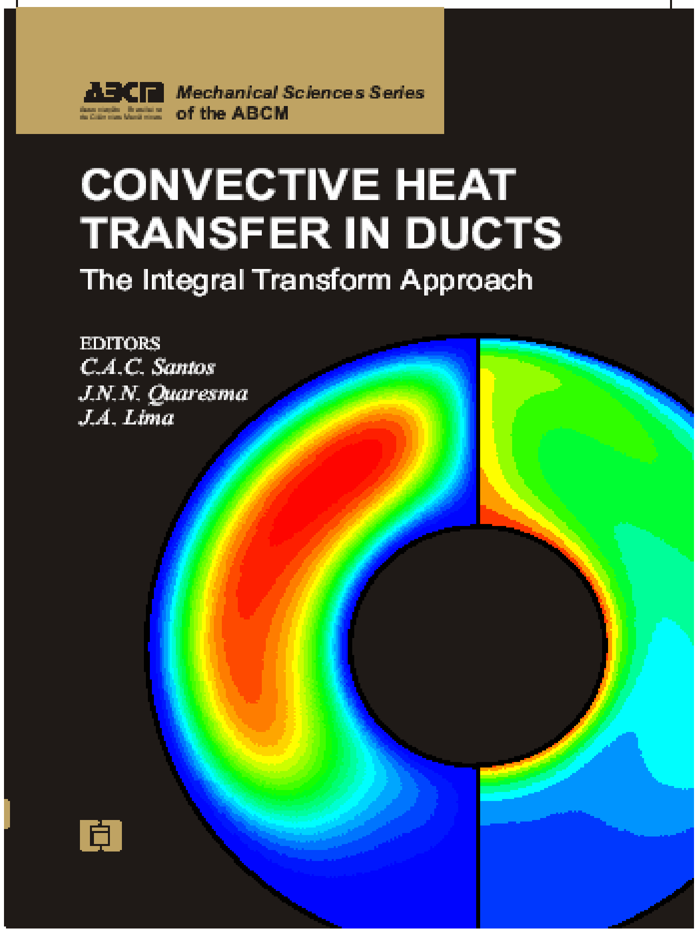
- Time-consuming numerical task is always in one single independent variable (ODE's).
- Reasonably simple computational implementation (subroutines libraries).
- Handles irregular domains directly.
- Automatic global error control.
- Mild increase in computational cost for increasing number of space variables.

Benchmarks

(Present Paper)

- Fluid flow and heat transfer inside channels – Review Paper (Navier-Stokes Eqs.)

Cotta, R.M., C.A.C. Santos, J.N.N. Quaresma, and J.S. Perez-Guerrero, "Hybrid Integral Transforms in Convection-Diffusion: Recent Applications in Internal Flow Simulation", Invited Lecture, Proc. of the **4th Int. Conf. Computational Heat and Mass Transfer, 4th ICCHMT**, Paris-Cachan, France, May 2005.



GITT
Problem Formulation

$$w_k(\boldsymbol{x})L_{t,k}T_k(\boldsymbol{x},t) + \boldsymbol{u}(\boldsymbol{x},t,T_\ell).\nabla T_k(\boldsymbol{x},t) + L_k T_k(\boldsymbol{x},t) =$$

$$P_k(\boldsymbol{x},t,T_\ell), \quad \boldsymbol{x} \in V, \quad t_0 < t < t_1, \quad k,\ell = 1,2,\ldots,n$$

;

$$L_{t,k} \equiv \frac{\partial}{\partial t} \quad or \quad L_{t,k} \equiv -a_k(t)\frac{\partial}{\partial t}\left[b_k(t)\frac{\partial}{\partial t}\right]$$

$$L_k \equiv -\nabla K_k(\boldsymbol{x})\nabla + d_k(\boldsymbol{x})$$

GITT
Boundary/Initial Conditions

$$T_k(\boldsymbol{x}, t_0) = f_k(\boldsymbol{x}), \quad \boldsymbol{x} \in V$$

or

$$\left[\delta_{k,p} - (-1)^p \gamma_{k,p} \frac{\partial}{\partial t}\right] T_k(\boldsymbol{x}, t_p) = f_{k,p}(\boldsymbol{x}, t_p, T_\ell)$$

$\boldsymbol{x} \in V, \quad p = 0,1$

;

$$\left[\alpha_k(\boldsymbol{x}) + \beta_k(\boldsymbol{x}) K_k(\boldsymbol{x}) \frac{\partial}{\partial \boldsymbol{n}}\right] T_k(\boldsymbol{x}, t) = \phi_k(\boldsymbol{x}, t, \mathrm{T}_\ell),$$

$$\boldsymbol{x} \in S, \quad \mathrm{t} > 0$$

GITT
Eigenvalue Problem & Transform Pair

$$L_k \psi_{ki}(\boldsymbol{x}) = \mu_{ki}^2 w_k(\boldsymbol{x}) \psi_{ki}(\boldsymbol{x}), \qquad \boldsymbol{x} \in V$$

$$\left[\alpha_k(\boldsymbol{x}) + \beta_k(\boldsymbol{x}) K_k(\boldsymbol{x}) \frac{\partial}{\partial \boldsymbol{n}}\right] \psi_{ki}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in S$$

$$\bar{T}_{k,i}(t) = \int_{\mathrm{v}} \mathrm{w}_{\mathrm{k}}(\boldsymbol{x}) \tilde{\psi}_{ki}(\boldsymbol{x}) T_k(\boldsymbol{x}, \mathrm{t}) \mathrm{dv}$$

Transforms

$$T_k(\boldsymbol{x}, t) = \sum_{i=1}^{\infty} \tilde{\psi}_{ki}(\boldsymbol{x}) \bar{T}_{k,i}(t)$$

Inverses

$$\tilde{\psi}_{ki}(\boldsymbol{x}) = \frac{\psi_{ki}(\boldsymbol{x})}{N_{ki}^{1/2}} \qquad N_{ki} = \int_{\mathrm{v}} w_k(\boldsymbol{x}) \psi_{ki}^2(\boldsymbol{x}) dv$$

GITT

Transformed System

Applying the operator $\int_{\mathrm{v}} \tilde{\psi}_{ki}(\boldsymbol{x})\mathrm{dv}$

$$L_{t,k}\overline{T}_{k,i}(t) + \sum_{j=1}^{\infty} a_{kij}(t,T_{\ell})\overline{T}_{\mathrm{k,j}}(t) = \overline{g}_{ki}(t,T_{l}),$$

$$\mathrm{i} = 1,2,..., \qquad \mathrm{t} > 0, \quad \mathrm{k}, \ell = 1,2,...,n$$

;

$$a_{kij}(t,T_{\ell}) = \delta_{ij}\mu_{ki}^{2} + a_{kij}^{*}(t,T_{\ell}), \; a_{kij}^{*}(t,T_{\ell}) = \int_{\mathrm{v}} \tilde{\psi}_{ki}(\boldsymbol{x})[\boldsymbol{u}(\boldsymbol{x},t,T_{\ell}).\nabla\,\tilde{\psi}_{ki}(\boldsymbol{x})]dv$$

$$\overline{g}_{ki}(t,T_{l}) = \int_{v} \tilde{\psi}_{ki}(\boldsymbol{x})P_{k}(\boldsymbol{x},t,\mathrm{T}_{\ell})dv + \int_{\mathrm{S}} \mathrm{K}_{\mathrm{k}}(\boldsymbol{x})$$

$$\left[\tilde{\psi}_{ki}(\boldsymbol{x})\frac{\partial T_{k}(\boldsymbol{x},t)}{\partial \boldsymbol{n}} - T_{\mathrm{k}}(\boldsymbol{x},t)\frac{\partial \tilde{\psi}_{ki}(\boldsymbol{x})}{\partial \boldsymbol{n}}\right]\mathrm{ds}$$

GITT
Transformed initial/boundary cond.

Applying the operator $\int_{v} \mathrm{w}_{k}(\boldsymbol{x})\tilde{\psi}_{ki}(\boldsymbol{x})dv$

;

$$\bar{T}_{k,i}(t_0) = \bar{f}_{ki} \equiv \int_{v} \mathrm{w}_{k}(\boldsymbol{x})\tilde{\psi}_{ki}(\boldsymbol{x}) f_k(\boldsymbol{x})dv$$

or

$$\left[\delta_{k,p} - (-1)^p \gamma_{k,p} \frac{\partial}{\partial t}\right]\bar{T}_{k,i}(t_p) = \bar{f}_{k,pi} \equiv$$

$$\int_{v} w_k(\boldsymbol{x})\tilde{\psi}_{ki}(\boldsymbol{x}) f_{k,p}(\boldsymbol{x}, t_p, T_\ell)dv$$

GITT
Convergence & Filtering

Filtering

$$T_k(\boldsymbol{x},t) = \theta_k(\boldsymbol{x},t) + T_{f,k}(\boldsymbol{x};t)$$

;

Convergence Testing

$$\varepsilon = \max_{\boldsymbol{x} \in V} \left| \frac{\sum_{i=N^*}^{N} \tilde{\psi}_{ki}(\boldsymbol{x}) \bar{T}_{k,i}(t)}{T_{f,k}(\boldsymbol{x};t) + \sum_{i=1}^{N} \tilde{\psi}_{ki}(\boldsymbol{x}) \bar{T}_{k,i}(t)} \right|$$

Total Transformation

PARABOLIC & PARABOLIC-HYPERBOLIC:

1 D- 3 D PDE $\xrightarrow{\int}$ **System of ODE's (IVP)**

DIVPAG/IMSL, NDSolve

ELLIPTIC:

2 D- 3 D PDE $\xrightarrow{\int}$ **System of ODE's (BVP)**

DBVPFD/IMSL

Partial Transformation

PARABOLIC & PARABOLIC-HYPERBOLIC:

2 D- 3 D PDE $\xrightarrow{\int}$ **1D – Systemof PDE's (DMOLCH/IMSL, NDSolve)**

ELLIPTIC:

3 D PDE $\xrightarrow{\int}$ **2D – System of PDE's**

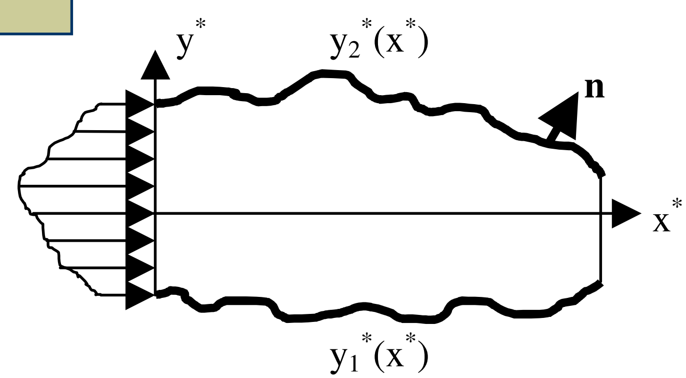
Streamfunction Formulation

Navier-Stokes Equations

$$\frac{\partial \psi}{\partial y}\left(\frac{\partial^3 \psi}{\partial x^3}+\frac{\partial^3 \psi}{\partial x \partial y^2}\right)-\frac{\partial \psi}{\partial x}\left(\frac{\partial^3 \psi}{\partial x^2 \partial y}+\frac{\partial^3 \psi}{\partial y^3}\right)=$$

$$\frac{1}{Re}\left(\frac{\partial^4 \psi}{\partial x^4}+2\frac{\partial^4 \psi}{\partial x^2 \partial y^2}+\frac{\partial^4 \psi}{\partial y^4}\right)$$

$$u=\frac{\partial \psi}{\partial y} \qquad v=-\frac{\partial \psi}{\partial x}$$



General irregular geometry and coordinates system for channel flow.

Primitive Variables Formulation

Navier-Stokes Equations

$$\frac{\partial U(X,Y)}{\partial X} + \frac{\partial V(X,Y)}{\partial Y} = 0, \qquad X > 0, \quad 0 < Y < 1$$

$$U\frac{\partial U(X,Y)}{\partial X} + V\frac{\partial U(X,Y)}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right), \quad X > 0, \quad 0 < Y < 1$$

$$\frac{\partial^2 P(X,Y)}{\partial X^2} + \frac{\partial^2 P(X,Y)}{\partial Y^2} = 2\left[\frac{\partial U}{\partial X}\frac{\partial V}{\partial Y} - \frac{\partial V}{\partial X}\frac{\partial U}{\partial Y}\right], \quad X > 0, \quad 0 < Y < 1$$

$$\frac{\partial P(X,1)}{\partial Y} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 V}{\partial Y^2}\right)_{Y=1}$$

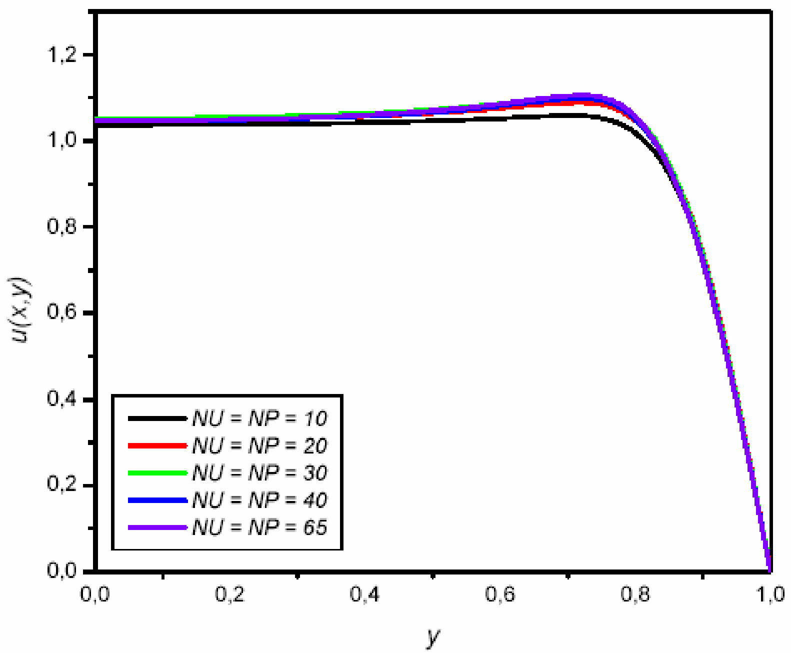
S.F.versus P.V. Formulations
Regular Channel

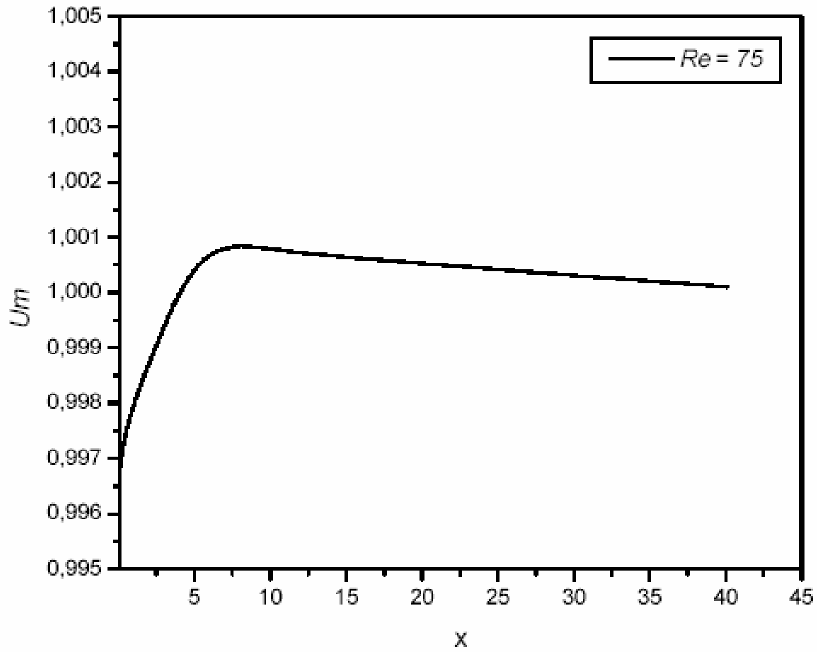
Table 1 – Covalidation of centerline velocity along channel length, *U(X,0)*, between the primitive variables [27] and the streamfunction [18] formulations.

Re	X Form.	0.2083	3.3333	7.5000
75	P.V.	1.050	1.334	1.444
75*	S.F.	1.052	1.337	1.444
150	P.V.	1.039	1.243	1.348
150*	S.F.	1.036	1.242	1.347
300	P.V.	1.026	1.173	1.252
300*	S.F.	1.024	1.170	1.250

Primitive Variables, P.V., Ref. [27]
(*) Streamfunction S.F., Ref.[18] : ***Re*=4.Re*** **- Relative error control 10^{-4}**

Primitive Variables Formulation
Parallel Plates Channel





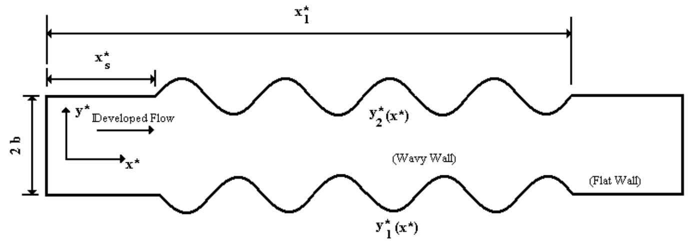
Primitive Variables Formulation

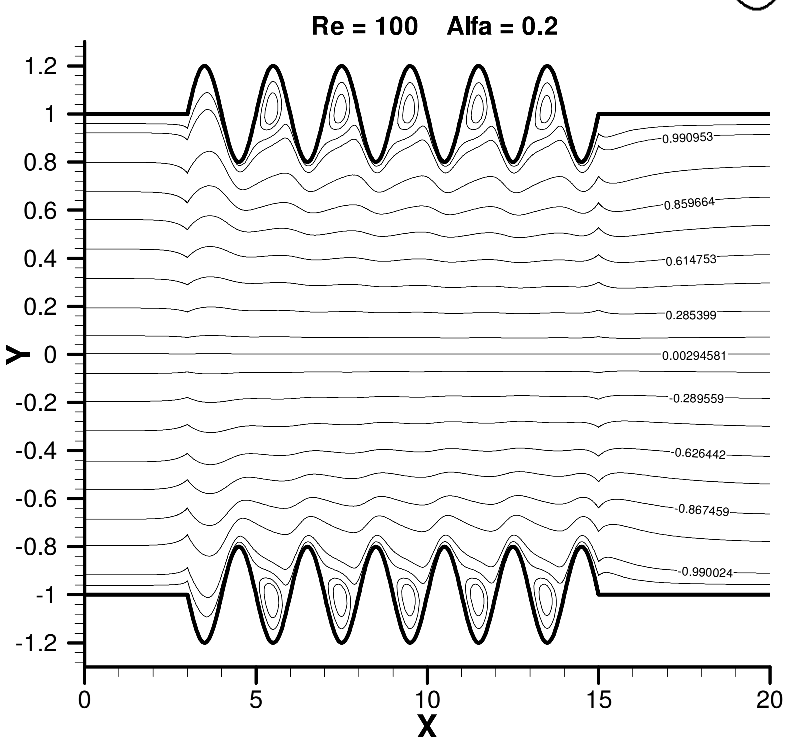
Convergence behavior of centerline velocity along channel length, *U(X,0)*, for *NU=NP* and *Re*=300 (Re*=1200).

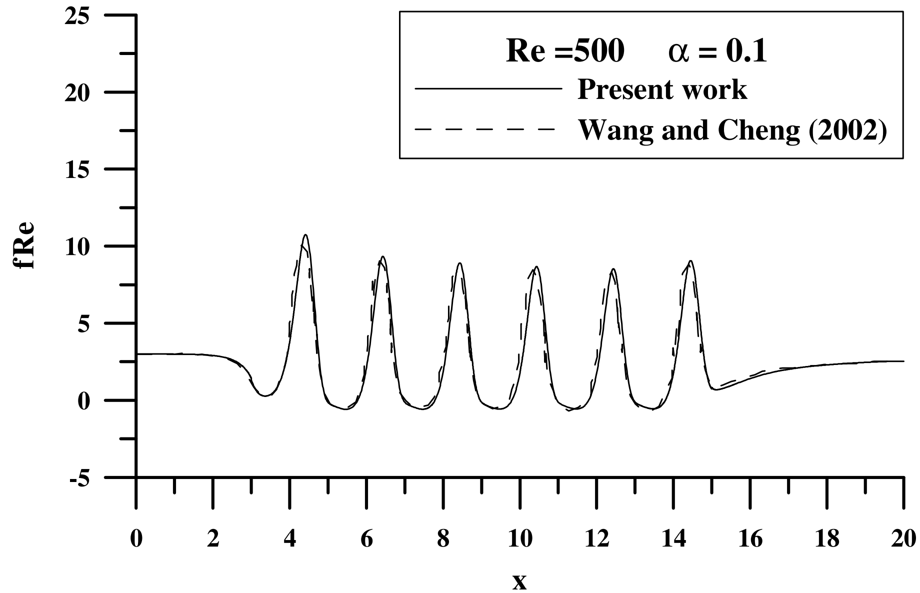
	X				
NU=NP	**0.20833**	**0.8333**	**3.3333**	**7.5**	**13.275**
5	1.006	1.053	1.113	1.200	1.281
10	1.004	1.054	1.143	1.223	1.304
15	1.010	1.063	1.153	1.233	1.311
20	1.014	1.068	1.157	1.237	1.314
25	1.016	1.070	1.160	1.240	1.317
30	1.018	1.071	1.161	1.244	1.324
35	1.020	1.074	1.163	1.247	1.327
40	1.021	1.076	1.165	1.248	1.330
45	1.022	1.077	1.167	1.249	1.333
50	1.023	1.078	1.169	1.250	1.337
55	1.024	1.080	1.171	1.252	1.340
60	1.025	1.082	1.172	1.252	1.341
65	1.026	1.083	1.173	1.252	1.341
70	1.026	1.083	1.173	1.252	1.341

Streamfunction Formulation

Irregular Channel







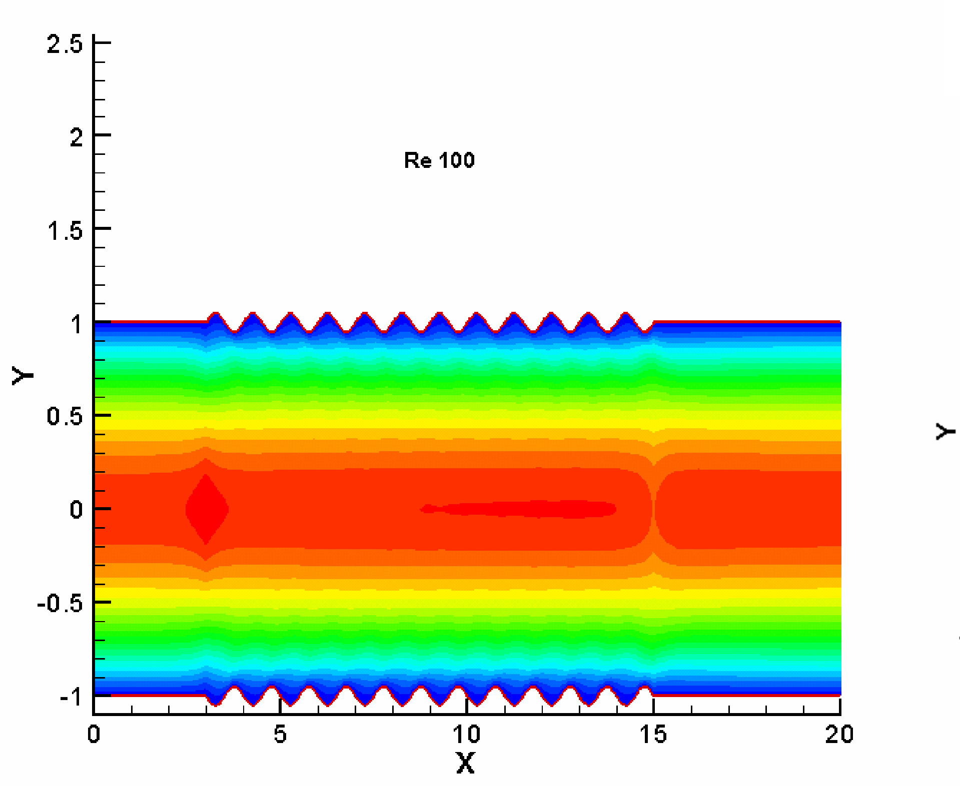
Streamfunction Formulation

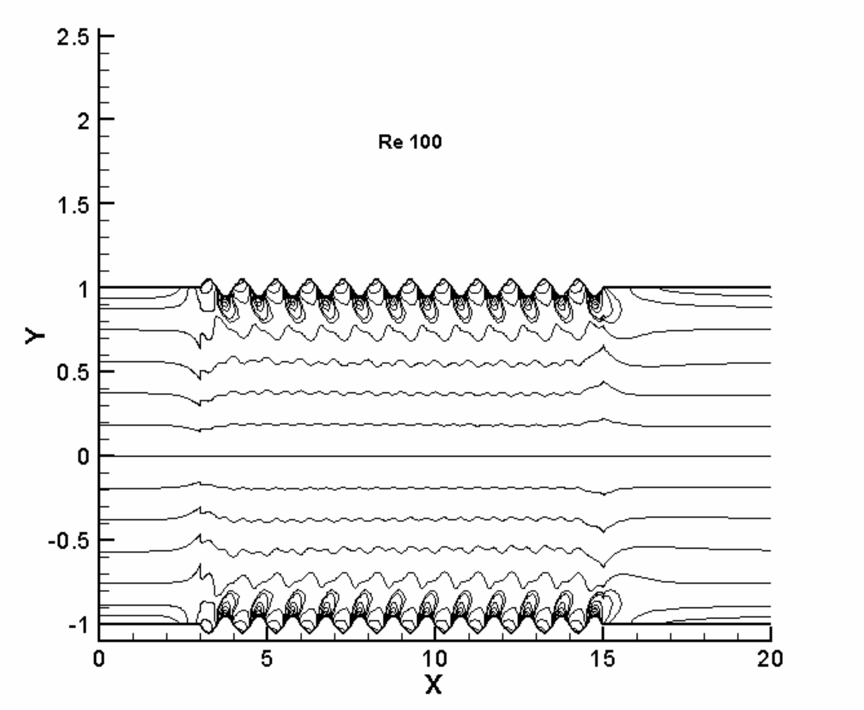
Convergence Behavior of the Streamfunction at $y = 0.5$ for $Re = 100$ and $\alpha = 0.2$.

x	N = 6	N = 10	N = 14	N = 18	N = 30
3	0.7321	0.7326	0.7327	0.7327	0.7327
5	0.7312	0.7380	0.7390	0.7391	0.7391
7	0.7470	0.7523	0.7531	0.7532	0.7532
9	0.7539	0.7588	0.7595	0.7595	0.7593
11	0.7577	0.7624	0.7630	0.7631	0.7631
13	0.7599	0.7646	0.7652	0.7652	0.7652
15	0.7167	0.7204	0.7211	0.7212	0.7212
20	0.7147	0.7157	0.7159	0.7159	0.7159

Streamfunction Formulation

Rough Channel Simulation: Streamfunction at *y* = 0.5 for *Re* = 100 ,
$\alpha = 5\%$ and $\omega = 2\pi$





Streamfunction Formulation

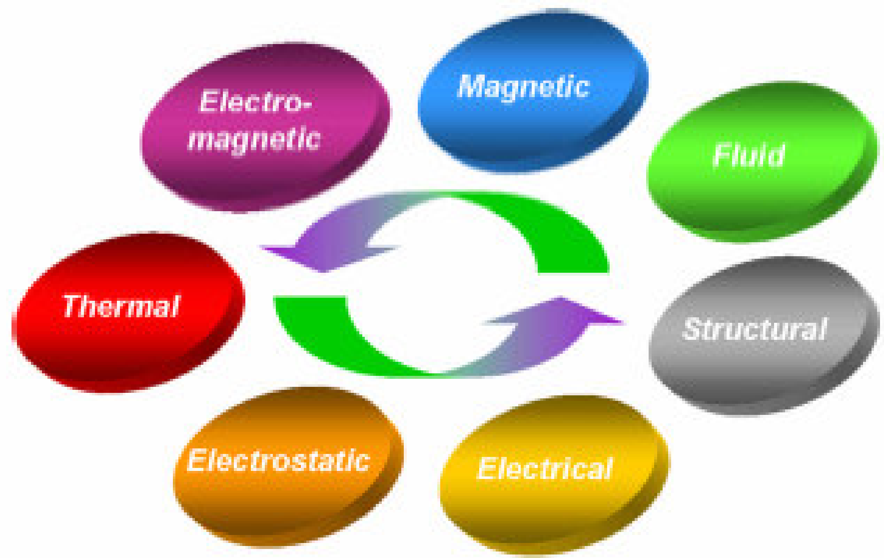
Rough Channel Simulation: Streamfunction at $y = 0.5$ for $Re = 100$, $\alpha = 5\%$ and $\omega = 2\pi$

NTV	x = 0	x = 3	x = 9	x = 15	x = 18
6	.68750	.71648	.69453	.66659	.69096
10	.68750	.71677	.69537	.66710	.69107
14	.68750	.71681	.69568	.66733	.69112
18	.68750	.71681	.69570	.66736	.69114

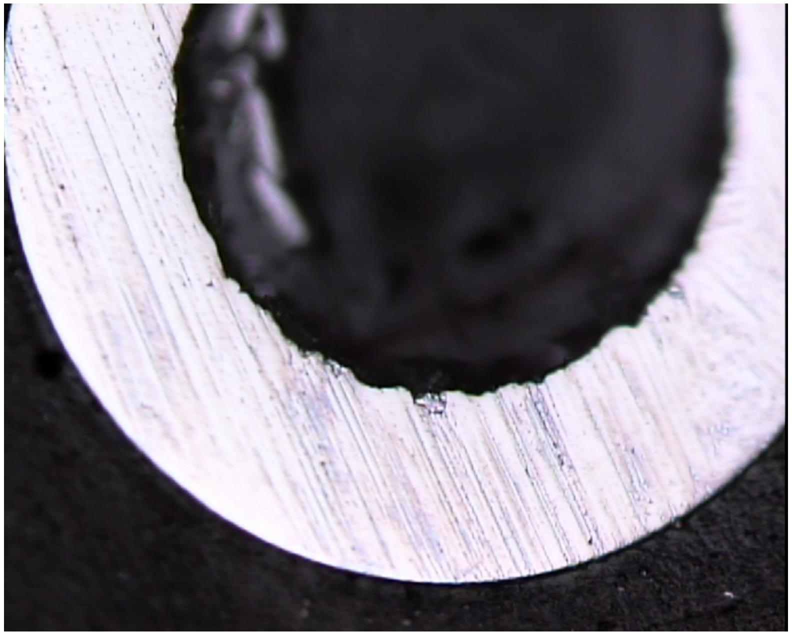
Present Research LTTC/COPPE/UFRJ

- *Millenium Institute*
  Hybrid Methods in Engineering and Multiphysics

137 Researchers
34 Institutions
30 International Collaborators
R$ 1.880.000,00
(3 years – US$ 750,000.00)



Present Research
Millenium Institute



I. Compilation and organization of available codes and developments

II. UNIT Code design and construction

III. Hybrid solutions for engineering problems with multiphysics

- Environmental Modeling
- Micro-Electro-Mechanical Systems (MEMS)
- Nano-structured Materials (Solids and Fluids)
- Emerging Energy Sources (Natural Gas, Hydrogen and Nuclear Energy)
- Bioengineering Modeling (Biofluids, Bioheat, and Tissue Engineering)

Other initiatives LTTC/COPPE/UFRJ

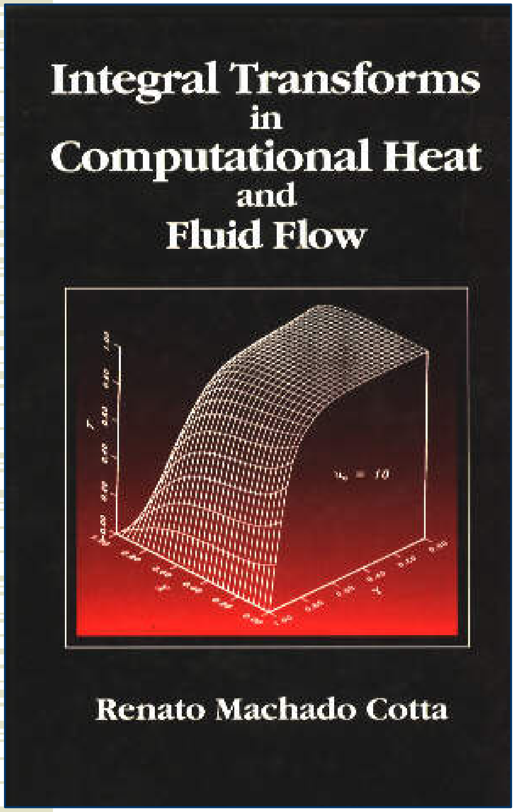
- *Mathematica* Technical Center –

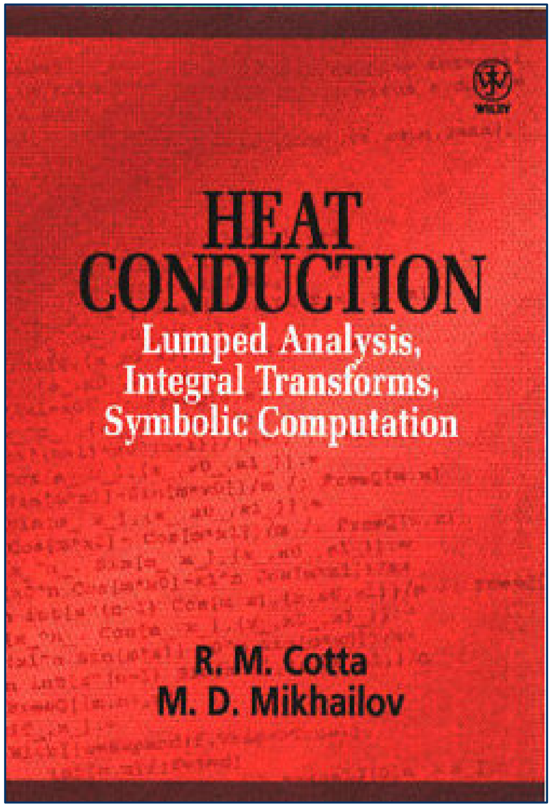
PEM/COPPE & Wolfram Research Inc., USA

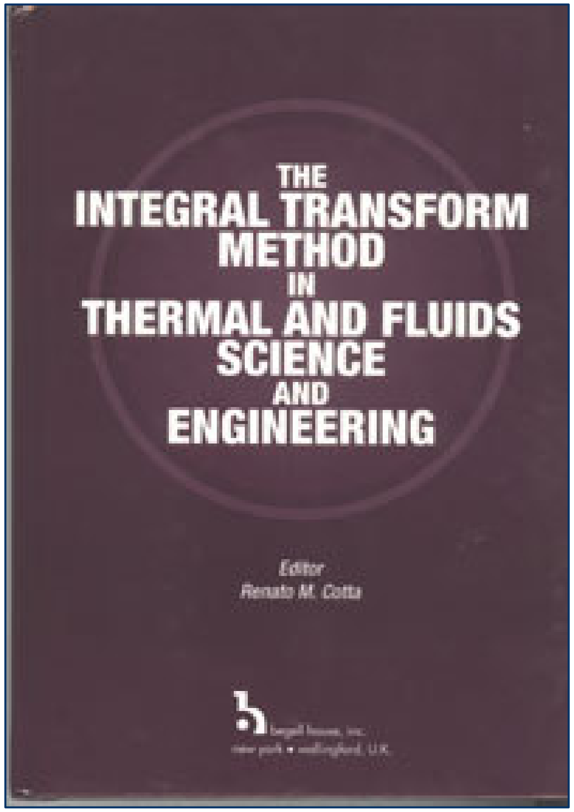
MTC is responsible for basic and advanced training and consulting in mixed symbolic-numerical computation with the platform *Mathematica*.

Sources

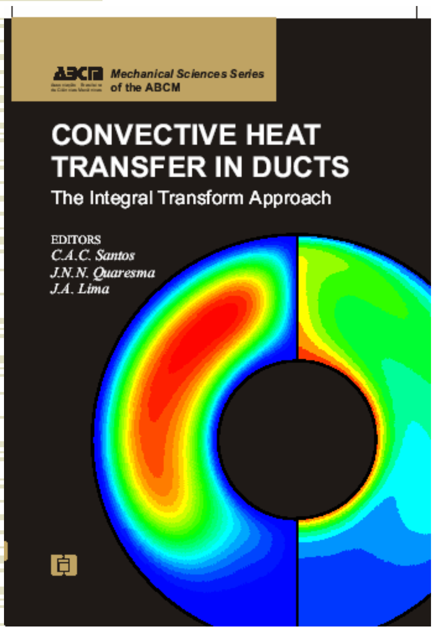
Books and Journal

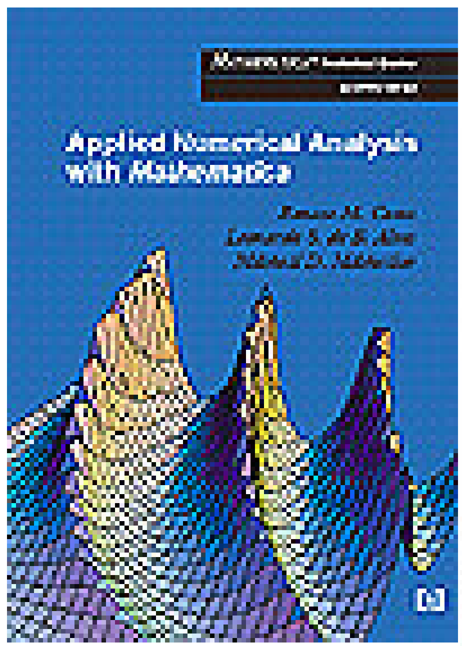


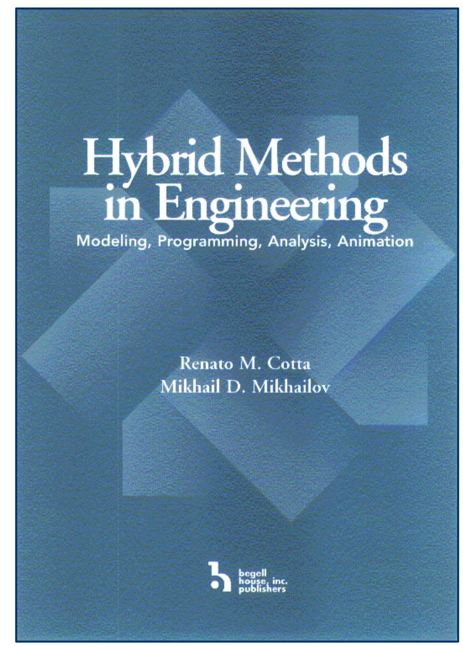




Sources
Books and Journal







New Book CASEE/COPPE/UFRJ

- **HYBRID METHODS IN ENVIRONMENTAL TRANSPORT PHENOMENA**

Renato M. Cotta, Robert Goldstein,

Paulo F. L. Heilbron, Michael J. Ungs

Editors

CASEE

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