Hybrid Integral Transforms in Convection-Diffusion

Renato M. Cotta, Francesco Scofano Neto* & Rodrigo Guedes*

Laboratory of Transmission & Technology of Heat - LTTC
PEM – COPPE/UFRJ, Brasil
(*) Instituto Militar de Engenharia, Rio de Janeiro, Brasil

PROPFIS
Escola Sul-Americana em Identificação de Propriedades Físicas em Transferência de Calor e Massa
Rio de Janeiro, Junho de 2005
The Hybrid Approach

The Simulation Process

- Real Problem
- Physical Model
  - Mathematical Model (C.I.E.A.)
  - Solution Methodology (G.I.T.T.)
  - Inverse Problem (Identification & Design)
    - Algorithms Implementation (IMSL & Mathematica)
    - Results Interpretation
Motivation

- Develop improved lumped-differential formulations in heat and fluid flow.
- Advance a hybrid numerical-analytical solution methodology for PDE’s.
- Exploit new concepts on algorithm implementation, based on mixed symbolic-numerical computation.
- Construct new algorithms for inverse problem analysis based on such hybrid paths.
Hybrid Tools

- The Coupled Integral Equations Approach (Improved Formulations).
- The Generalized Integral Transform Technique (Hybrid Methods).
- The *Mathematica* System (Mixed Computations)
- Inverse Problems (Identification & Design)
Hybrid Numerical-Analytical Methods

The Generalized Integral Transform Technique

GITT
Goals

- Reduce computational costs in the solution of multidimensional PDE’s, with respect to classical discrete approaches.
- Offer automatic global accuracy control, allowing for the establishment of a benchmarks database in heat and fluid flow.
- Fully exploit the analytic nature of a hybrid approach for implementation in mixed symbolic-numerical computation platforms.
The Generalized Integral Transform Technique - GITT

- Choose the associated eigenvalue problem.
- Develop the integral transform pair.
- Integral transform the original PDE.
- Numerically (or analytically) solve the resulting coupled ODE system for the transformed potentials.
- Recall the analytical inversion formula to reconstruct the hybrid solution of the desired potential.
Classes of Problems
(Linear and Nonlinear)

- Diffusion
- Convection-Diffusion
- Eigenvalue Problems
- Boundary Layer Equations
- Navier-Stokes Equations
Advantages

- Time-consuming numerical task is always in one single independent variable (ODE’s).
- Reasonably simple computational implementation (subroutines libraries).
- Handles irregular domains directly.
- Automatic global error control.
- Mild increase in computational cost for increasing number of space variables.
Benchmarks

(Present Paper)

- Fluid flow and heat transfer inside channels – Review Paper (Navier-Stokes Eqs.)

Problem Formulation

\[ w_k(x)L_{t,k}T_k(x,t) + u(x,t,T_\ell) \nabla T_k(x,t) + L_kT_k(x,t) = \]

\[ P_k(x,t,T_\ell), \quad x \in V, \quad t_0 < t < t_1, \quad k, \ell = 1, 2, \ldots, n \]

\[ L_{t,k} \equiv \frac{\partial}{\partial t} \quad \text{or} \quad L_{t,k} \equiv -a_k(t) \frac{\partial}{\partial t} \left[ b_k(t) \frac{\partial}{\partial t} \right] \]

\[ L_k \equiv -\nabla K_k(x) \nabla + d_k(x) \]
<table>
<thead>
<tr>
<th>GITT Boundary/Initial Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_k(x, t_0) = f_k(x), \quad x \in V$</td>
</tr>
<tr>
<td>or</td>
</tr>
</tbody>
</table>
| \[
\begin{bmatrix}
\delta_{k,p} - (-1)^p \gamma_{k,p} \frac{\partial}{\partial t}
\end{bmatrix} T_k(x, t_p) = f_{k,p}(x, t_p, T_\ell) \quad x \in V, \quad p = 0, 1
\] |
| $[\begin{bmatrix}
\alpha_k(x) + \beta_k(x) K_k(x) \frac{\partial}{\partial n}
\end{bmatrix} T_k(x, t) = \phi_k(x, t, T_\ell), \quad x \in S, \quad t > 0$ |
GITT

Eigenvalue Problem & Transform Pair

\[ L_k \psi_{ki}(x) = \mu_k^2 w_k(x) \psi_{ki}(x), \quad x \in V \]

\[ \left[ \alpha_k(x) + \beta_k(x) K_k(x) \frac{\partial}{\partial n} \right] \psi_{ki}(x) = 0, \quad x \in S \]

\[ \bar{T}_{k,i}(t) = \int_v w_k(x) \bar{\psi}_{ki}(x) T_k(x,t) dv \quad \text{Transforms} \]

\[ T_k(x,t) = \sum_{i=1}^{\infty} \bar{\psi}_{ki}(x) \bar{T}_{k,i}(t) \quad \text{Inverses} \]

\[ \bar{\psi}_{ki}(x) = \frac{\psi_{ki}(x)}{N_{ki}^{1/2}} \]

\[ N_{ki} = \int_v w_k(x) \psi_{ki}^2(x) dv \]
Applying the operator \[ \int_v \tilde{\psi}_{ki}(x) dv \]

\[ L_{t,k} \tilde{T}_{k,i}(t) + \sum_{j=1}^{\infty} a_{kj}(t, T_\ell) \tilde{T}_{k,j}(t) = \overline{g}_{ki}(t, T_\ell), \]

\[ i = 1, 2, \ldots, \quad t > 0, \quad k, \ell = 1, 2, \ldots, n \]

\[ a_{kj}(t, T_\ell) = \delta_{ij} \mu_{ki}^2 + a^*_{kj}(t, T_\ell), \quad a^*_{kj}(t, T_\ell) = \int_v \tilde{\psi}_{ki}(x)[u(x, t, T_\ell) \nabla \tilde{\psi}_{ki}(x)] dv \]

\[ \overline{g}_{ki}(t, T_\ell) = \int_v \tilde{\psi}_{ki}(x) P_k(x, t, T_\ell) dv + \int_S K_k(x) \]

\[ \left[ \tilde{\psi}_{ki}(x) \frac{\partial T_k(x, t)}{\partial n} - T_k(x, t) \frac{\partial \tilde{\psi}_{ki}(x)}{\partial n} \right] ds \]
GITI
Transformed initial/boundary cond.

Applying the operator
\[ \int_v w_k(x) \tilde{\psi}_{ki}(x) dv \]

\[ \tilde{T}_{k,i}(t_0) = \tilde{f}_{ki} \equiv \int_v w_k(x) \tilde{\psi}_{ki}(x) f_k(x) dv \]

or

\[ \left[ \delta_{k,p} - (-1)^p \gamma_{k,p} \frac{\partial}{\partial t} \right] \tilde{T}_{k,i}(t_p) = \tilde{f}_{k,pi} \equiv \int_v w_k(x) \tilde{\psi}_{ki}(x) f_{k,p}(x,t_p,T_\ell) dv \]
GIT\text{T}

Convergence & Filtering

Filtering

\[ T_k(x,t) = \theta_k(x,t) + T_{f,k}(x;t) \]

Convergence Testing

\[ \mathcal{E} = \max_{x \in V} \left| \frac{\sum_{i=N^*}^{N} \widetilde{\psi}_{ki}(x)\overline{T}_{k,i}(t)}{T_{f,k}(x;t) + \sum_{i=1}^{N} \widetilde{\psi}_{ki}(x)\overline{T}_{k,i}(t)} \right| \]
Total Transformation

**PARABOLIC & PARABOLIC-HYPERBOLIC:**

1 D- 3 D PDE \[\int\] System of ODE’s (IVP)
DIVPAG/IMSL, NDSolve

**ELLIPTIC:**

2 D- 3 D PDE \[\int\] System of ODE’s (BVP)
DBVPFD/IMSL
Partial Transformation

PARABOLIC & PARABOLIC-HYPERBOLIC:

2 D- 3 D PDE $\xrightarrow{\int} 1D – \text{System of PDE’s (DMOLCH/IMSL, NDSolve)}$

ELLiptic:

3 D PDE $\xrightarrow{\int} 2D – \text{System of PDE’s}$
Streamfunction Formulation

Navier-Stokes Equations

\[ \frac{\partial \psi}{\partial y} \left( \frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^3 \psi}{\partial x \partial y^2} \right) - \frac{\partial \psi}{\partial x} \left( \frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial y^3} \right) = \]

\[ \frac{1}{Re} \left( \frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} \right) \]

\[ u = \frac{\partial \psi}{\partial y} \quad v = - \frac{\partial \psi}{\partial x} \]

General irregular geometry and coordinates system for channel flow.
Primitive Variables Formulation
Navier-Stokes Equations

\[ \frac{\partial U(X,Y)}{\partial X} + \frac{\partial V(X,Y)}{\partial Y} = 0, \quad X > 0, \quad 0 < Y < 1 \]

\[ U \frac{\partial U(X,Y)}{\partial X} + V \frac{\partial U(X,Y)}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{1}{\text{Re}} \left( \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right), \quad X > 0, \quad 0 < Y < 1 \]

\[ \frac{\partial^2 P(X,Y)}{\partial X^2} + \frac{\partial^2 P(X,Y)}{\partial Y^2} = 2 \left[ \frac{\partial U}{\partial X} \frac{\partial V}{\partial Y} - \frac{\partial V}{\partial X} \frac{\partial U}{\partial Y} \right], \quad X > 0, \quad 0 < Y < 1 \]

\[ \frac{\partial P(X,1)}{\partial Y} = \frac{1}{\text{Re}} \left( \frac{\partial^2 V}{\partial Y^2} \right)_{Y=1} \]
### S.F. versus P.V. Formulations

#### Regular Channel

Table 1 – Covalidation of centerline velocity along channel length, \( U(X,0) \), between the primitive variables [27] and the streamfunction [18] formulations.

<table>
<thead>
<tr>
<th>Re</th>
<th>X Form.</th>
<th>(0.2083)</th>
<th>(3.3333)</th>
<th>(7.5000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>P.V.</td>
<td>1.050</td>
<td>1.334</td>
<td>1.444</td>
</tr>
<tr>
<td>75*</td>
<td>S.F.</td>
<td>1.052</td>
<td>1.337</td>
<td>1.444</td>
</tr>
<tr>
<td>150</td>
<td>P.V.</td>
<td>1.039</td>
<td>1.243</td>
<td>1.348</td>
</tr>
<tr>
<td>150*</td>
<td>S.F.</td>
<td>1.036</td>
<td>1.242</td>
<td>1.347</td>
</tr>
<tr>
<td>300</td>
<td>P.V.</td>
<td>1.026</td>
<td>1.173</td>
<td>1.252</td>
</tr>
<tr>
<td>300*</td>
<td>S.F.</td>
<td>1.024</td>
<td>1.170</td>
<td>1.250</td>
</tr>
</tbody>
</table>

**Primitive Variables, P.V., Ref. [27]**

(*) **Streamfunction S.F., Ref.[18]**: \( Re^* = 4.Re \) - Relative error control \( 10^{-4} \)
Primitive Variables Formulation
Parallel Plates Channel
## Primitive Variables Formulation

Convergence behavior of centerline velocity along channel length, $U(X,0)$, for $NU=NP$ and $Re=300$ ($Re^*=1200$).

<table>
<thead>
<tr>
<th>NU=NP</th>
<th>0.20833</th>
<th>0.8333</th>
<th>3.3333</th>
<th>7.5</th>
<th>13.275</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.006</td>
<td>1.053</td>
<td>1.113</td>
<td>1.200</td>
<td>1.281</td>
</tr>
<tr>
<td>10</td>
<td>1.004</td>
<td>1.054</td>
<td>1.143</td>
<td>1.223</td>
<td>1.304</td>
</tr>
<tr>
<td>15</td>
<td>1.010</td>
<td>1.063</td>
<td>1.153</td>
<td>1.233</td>
<td>1.311</td>
</tr>
<tr>
<td>20</td>
<td>1.014</td>
<td>1.068</td>
<td>1.157</td>
<td>1.237</td>
<td>1.314</td>
</tr>
<tr>
<td>25</td>
<td>1.016</td>
<td>1.070</td>
<td>1.160</td>
<td>1.240</td>
<td>1.317</td>
</tr>
<tr>
<td>30</td>
<td>1.018</td>
<td>1.071</td>
<td>1.161</td>
<td>1.244</td>
<td>1.324</td>
</tr>
<tr>
<td>35</td>
<td>1.020</td>
<td>1.074</td>
<td>1.163</td>
<td>1.247</td>
<td>1.327</td>
</tr>
<tr>
<td>40</td>
<td>1.021</td>
<td>1.076</td>
<td>1.165</td>
<td>1.248</td>
<td>1.330</td>
</tr>
<tr>
<td>45</td>
<td>1.022</td>
<td>1.077</td>
<td>1.167</td>
<td>1.249</td>
<td>1.333</td>
</tr>
<tr>
<td>50</td>
<td>1.023</td>
<td>1.078</td>
<td>1.169</td>
<td>1.250</td>
<td>1.337</td>
</tr>
<tr>
<td>55</td>
<td>1.024</td>
<td>1.080</td>
<td>1.171</td>
<td>1.252</td>
<td>1.340</td>
</tr>
<tr>
<td>60</td>
<td>1.025</td>
<td>1.082</td>
<td>1.172</td>
<td>1.252</td>
<td>1.341</td>
</tr>
<tr>
<td>65</td>
<td>1.026</td>
<td>1.083</td>
<td>1.173</td>
<td>1.252</td>
<td>1.341</td>
</tr>
<tr>
<td>70</td>
<td>1.026</td>
<td>1.083</td>
<td>1.173</td>
<td>1.252</td>
<td>1.341</td>
</tr>
</tbody>
</table>
Streamfunction Formulation
Irregular Channel

Re = 100  \( \alpha = 0.2 \)

Re = 500  \( \alpha = 0.1 \)

Present work
Streamfunction Formulation

Convergence Behavior of the Streamfunction at $y = 0.5$ for $Re = 100$ and $\alpha = 0.2$.

<table>
<thead>
<tr>
<th>x</th>
<th>$N = 6$</th>
<th>$N = 10$</th>
<th>$N = 14$</th>
<th>$N = 18$</th>
<th>$N = 30$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.7321</td>
<td>0.7326</td>
<td>0.7327</td>
<td>0.7327</td>
<td>0.7327</td>
</tr>
<tr>
<td>5</td>
<td>0.7312</td>
<td>0.7380</td>
<td>0.7390</td>
<td>0.7391</td>
<td>0.7391</td>
</tr>
<tr>
<td>7</td>
<td>0.7470</td>
<td>0.7523</td>
<td>0.7531</td>
<td>0.7532</td>
<td>0.7532</td>
</tr>
<tr>
<td>9</td>
<td>0.7539</td>
<td>0.7588</td>
<td>0.7595</td>
<td>0.7595</td>
<td>0.7593</td>
</tr>
<tr>
<td>11</td>
<td>0.7577</td>
<td>0.7624</td>
<td>0.7630</td>
<td>0.7631</td>
<td>0.7631</td>
</tr>
<tr>
<td>13</td>
<td>0.7599</td>
<td>0.7646</td>
<td>0.7652</td>
<td>0.7652</td>
<td>0.7652</td>
</tr>
<tr>
<td>15</td>
<td>0.7167</td>
<td>0.7204</td>
<td>0.7211</td>
<td>0.7212</td>
<td>0.7212</td>
</tr>
<tr>
<td>20</td>
<td>0.7147</td>
<td>0.7157</td>
<td>0.7159</td>
<td>0.7159</td>
<td>0.7159</td>
</tr>
</tbody>
</table>
Streamfunction Formulation

Rough Channel Simulation: Streamfunction at $y = 0.5$ for $Re = 100$, $\alpha = 5\%$ and $\omega = 2\pi$
Streamfunction Formulation

Rough Channel Simulation: Streamfunction at $y = 0.5$ for $Re = 100$, $\alpha = 5\%$ and $\omega = 2\pi$

<table>
<thead>
<tr>
<th>NTV</th>
<th>$x = 0$</th>
<th>$x = 3$</th>
<th>$x = 9$</th>
<th>$x = 15$</th>
<th>$x = 18$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0.68750</td>
<td>0.71648</td>
<td>0.69453</td>
<td>0.66659</td>
<td>0.69096</td>
</tr>
<tr>
<td>10</td>
<td>0.68750</td>
<td>0.71677</td>
<td>0.69537</td>
<td>0.66710</td>
<td>0.69107</td>
</tr>
<tr>
<td>14</td>
<td>0.68750</td>
<td>0.71681</td>
<td>0.69568</td>
<td>0.66733</td>
<td>0.69112</td>
</tr>
<tr>
<td>18</td>
<td>0.68750</td>
<td>0.71681</td>
<td>0.69570</td>
<td>0.66736</td>
<td>0.69114</td>
</tr>
</tbody>
</table>
Present Research
LTTC/COPPE/UFRJ

- Millenium Institute
  Hybrid Methods in Engineering and Multiphysics

137 Researchers
34 Institutions
30 International Collaborators
R$ 1.880.000,00
(3 years – US$ 750,000.00)
Present Research
Millenium Institute

I. Compilation and organization of available codes and developments
II. UNIT Code design and construction
III. Hybrid solutions for engineering problems with multiphysics
   - Environmental Modeling
   - Micro-Electro-Mechanical Systems (MEMS)
   - Nano-structured Materials (Solids and Fluids)
   - Emerging Energy Sources (Natural Gas, Hydrogen and Nuclear Energy)
   - Bioengineering Modeling (Biofluids, Bioheat, and Tissue Engineering)
Other initiatives
LTTC/COPPE/UFRJ

- **Mathematica**
  Technical Center – PEM/COPPE & Wolfram Research Inc., USA

MTC is responsible for basic and advanced training and consulting in mixed symbolic-numerical computation with the platform Mathematica.
Sources
Books and Journal
Sources
Books and Journal

CONVETIVE HEAT TRANSFER IN DUCTS
The Integral Transform Approach

Applied Numerical Analysis with Mathematica

INVERSE HEAT TRANSFER

Hybrid Methods in Engineering
Modeling, Programming, Analysis, Animation
New Book
CASEE/COPPE/UFRJ

- HYBRID METHODS IN ENVIRONMENTAL TRANSPORT PHENOMENA

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